Phase Transition of a Scalar Field Theory in the Early Universe

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- 1-loop mass correction in expanding background
- application of WKB and its restrictions
- Numerical comparison

diploma thesis under supervision of

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Introduction

Scalar field theory with $\lambda\varphi^4$ self interaction in (thermal) real-time formalism

Expanding system, spatially flat FRW co-moving metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) dx^2 \qquad (1)$$

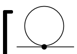
Propagators can be expressed in terms of mode functions which satisfy

$$\ddot{\tilde{u}}_{\vec{k}} + \left(\frac{k^2}{a^2} + m^2 - \frac{3}{4} \left(\frac{\dot{a}}{a} \right)^2 - \frac{3}{2} \frac{\ddot{a}}{a} \right) \tilde{u}_{\vec{k}} = 0 \qquad (2)$$

e.g. equal time propagator

$$G_c(t - t, \vec{k}) = \frac{i}{2a^3(t)} (1 + 2n_{\vec{k}}) \left| \tilde{u}_{\vec{k}}(t) \right|^2 \qquad (3)$$

Mass correction in curved space-time

Mass correction in $\lambda\varphi^4$-theory:
$m^2 = -\mu^2 + \delta m^2$ with $\delta m^2 = -i\lambda[$  $]$.

Proper renormalization $\Rightarrow$

$$m^2(t) = -\mu_R^2 + \frac{\lambda_R}{2\pi^2 a^3(t)} \times \int_0^\infty |\tilde{u}_k(t)|^2 \frac{1}{e^{\beta\omega_k(0)} - 1} k^2 dk \quad (4)$$

with mode functions

$$\ddot{\tilde{u}}_k + \left(\frac{k^2}{a^2} + m^2 - \frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a} \right) \tilde{u}_k = 0 \quad (5)$$

Coupled system of equations, numerically demanding. What can we do analytically? $\Rightarrow$ WKB

WKB solution and restrictions

General problem

$$\left(\partial_t^2 + \omega_{\vec{k}}^2(t)\right) u_{\vec{k}}(t) = 0 \qquad (6)$$

Ansatz for $\omega^2 > 0$

$$u_{\vec{k}}(t) = R(t) e^{iS(t)} \qquad (7)$$

gives $R = c/\sqrt{\dot{S}}$ and

$$\frac{3}{4}\dot{S}^{-2}\ddot{S}^2 - \frac{1}{2}\dot{S}^{-1}\dddot{S} - \dot{S}^2 + \omega^2 = 0 \qquad (8)$$

WKB approximation

$$u_{\vec{k}}(t) = \frac{c_{\vec{k}}}{\sqrt{\omega_{\vec{k}}(t)}} e^{\pm i \int_{t_0}^{t} dt' \omega_{\vec{k}}(t')} \qquad (9)$$

$$\boxed{\left|\frac{d}{dt}\left(\frac{1}{\omega_{\vec{k}}(t)}\right)\right| \ll 1 \quad \text{and} \quad \left|\frac{d^2}{dt^2}\left(\frac{1}{\omega_{\vec{k}}^2(t)}\right)\right| \ll 1} \qquad (10)$$

WKB in comoving metric

$$\omega_k^2(t) = \frac{k^2}{a^2} + m^2(t) - \frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a} \qquad (11)$$

leads to restrictions on $\bar{\omega}^2 = k^2/a^2 + m^2$

$$|H| = \left|\frac{\dot{a}}{a}\right| \ll \frac{\bar{\omega}}{4} \qquad \left|\frac{\ddot{a}}{a}\right| \ll \frac{\bar{\omega}^2}{2}$$

$$\left|\frac{\dot{m}}{m}\right| \ll \frac{\bar{\omega}}{4\beta} \qquad \left|\frac{\ddot{m}}{m}\right| \ll \frac{\bar{\omega}^2}{2\beta^2} \qquad (12)$$

with $1/\beta \equiv \bar{\omega}/m \geq 1$.

de Sitter universe: $a(t) \sim e^{Ht} \Rightarrow H \ll \frac{m(0)}{4}$

power law: $a(t) \sim (t+t_0)^p \Rightarrow t_0 \gg \frac{4p}{m(0)}$
($p = 1/2$ radiation dominated,
$p = 2/3$ matter dominated universe)

WKB for $\omega^2 < 0$

$$u_{\vec{k}}(t) = R(t)e^{S(t)} \tag{13}$$

$$u_{\vec{k}}(t) = \frac{c_{\vec{k}}}{\sqrt{\left|\omega_{\vec{k}}(t)\right|}} e^{\pm \int_{t_0}^{t} dt' \left|\omega_{\vec{k}}(t')\right|} \tag{14}$$

$$\left|\frac{d}{dt}\left(\frac{1}{\left|\omega_{\vec{k}}(t)\right|}\right)\right| \ll 1 \quad \text{and} \quad \left|\frac{d^2}{dt^2}\left(\frac{1}{\left|\omega_{\vec{k}}(t)\right|^2}\right)\right| \ll 1 \tag{15}$$

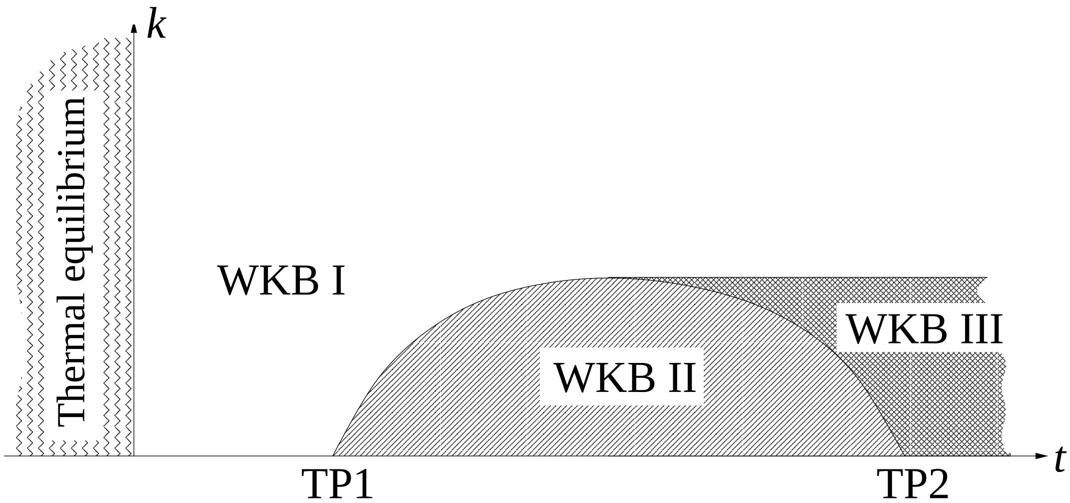
Link these solutions by *WKB connection formulae*

$$\left.\frac{1}{\sqrt{\omega}}\cos\left(\varphi + \frac{\pi}{4}\right)\right|_{t \ll t_1} \longleftrightarrow \left.\frac{1}{2}\frac{1}{\sqrt{|\omega|}}e^{-\varphi}\right|_{t \gg t_1} \tag{16}$$

$$\left.\frac{1}{\sqrt{\omega}}\sin\left(\varphi + \frac{\pi}{4}\right)\right|_{t \ll t_1} \longleftrightarrow \left.\frac{1}{\sqrt{|\omega|}}e^{\varphi}\right|_{t \gg t_1} \tag{17}$$

$\varphi = \varphi(t) = \int_{t_1}^{t} |\omega(t')|\, dt'$, $\omega = |\omega(t)|$

WKB approximation regions



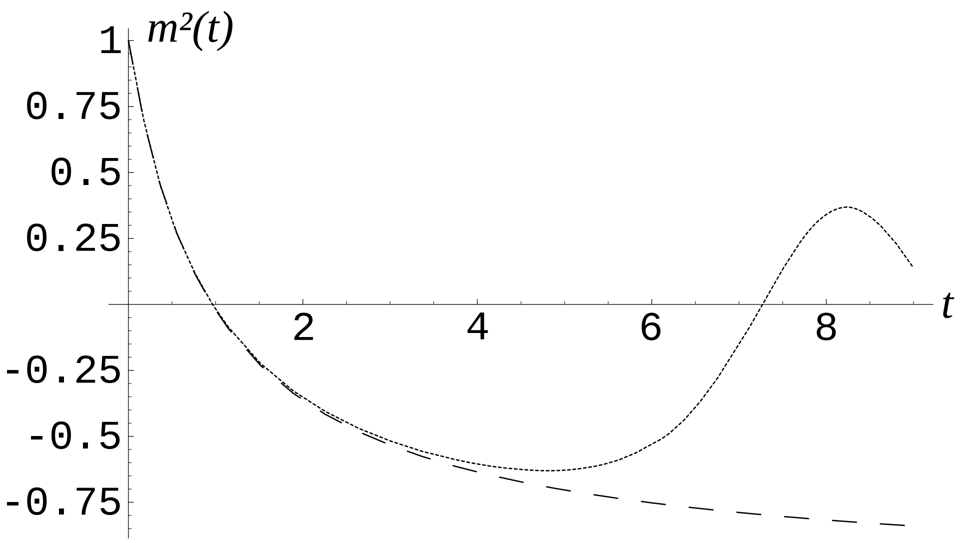
$m^2(t)$ changes sign at TP1 and TP2.

Region	$u_k(t)u_k^*(t)$
WKB I	$\dfrac{1}{2\omega_k(t)}$
WKB II	$\frac{1}{2\lvert\omega_k(t)\rvert}\left(e^{2\Omega_1(t)} + \frac{1}{4}e^{-2\Omega_1(t)}\right)$
WKB III	$\frac{1}{2\omega_k(t)}\left[\cosh(2Q) + \sinh(2Q)\cos(2\Omega_2(t))\right]$

$\Omega_1(t) \equiv \int_{\mathrm{TP1}}^{t} \lvert\omega_k(t')\rvert\, dt'$,
$\Omega_2(t) \equiv -\frac{\pi}{4} + \int_{\mathrm{TP2}}^{t} \omega_k(t') dt'$,
$Q \equiv \int_{\mathrm{TP1}}^{\mathrm{TP2}} \lvert\omega_k(t')\rvert\, dt' + \log 2$.

Taylor expansion for $T \gg m$ leads to simple formula for $t \lesssim$ TP1

$$\boxed{m^2(t) \approx -\mu_R^2 + \left(\mu_R^2 + m^2(0)\right)\left(\frac{a(0)}{a(t)}\right)^2} \tag{18}$$



Numerical result (full line) and analytical result (dashed line) for $m^2(t)$ in a radiation dominated universe ($\lambda_R = 0.01$, $1/T = 0.02$, $a(t) = \sqrt{t+1}$, $m(0) = 1$, $\mu_R^2 \approx 1.044$).

Solving for $m^2(\mathrm{TP1}) = 0 \Rightarrow$

$$\mathrm{TP1} \approx t_0 \frac{m^2(0)}{\mu_R^2} \approx 0.958 \tag{19}$$

(Numerical: TP1 = 0.964)

Mass Taylor expansion for general $\beta = 1/T$

$$\begin{aligned}
m_0^2 &= -\mu_R^2 + \frac{1}{\beta^2}\frac{\lambda_R}{2\pi^2}\mathcal{D}_{1,1,2} \\
m_0^{2\prime} &= -\frac{1}{\beta^2} H \frac{\lambda_R}{2\pi^2}\frac{\mathcal{D}_{2,2,4}}{1+\frac{1}{2}\frac{\lambda_R}{2\pi^2}\mathcal{D}_{3,1,2}} \\
m_0^{2\prime\prime} &= \frac{1}{\beta^2}\frac{\lambda_R}{2\pi^2}\frac{1}{1+\frac{1}{2}\frac{\lambda_R}{2\pi^2}\mathcal{D}_{3,1,2}} \\
&\quad \times \Big[H^2 \left(\mathcal{D}_{4,2,6} + 2\mathcal{D}_{3,3,6} - \mathcal{D}_{3,2,6} - \mathcal{D}_{2,2,4}\right) \\
&\quad\quad - H_2 \mathcal{D}_{2,2,4} + H\beta^2 m_0^{2\prime}\mathcal{D}_{4,2,4} + \frac{3}{4}\left(\beta^2 m_0^{2\prime}\right)^2 \mathcal{D}_{5,1,2}\Big]
\end{aligned}$$

with Debye functions

$$\boxed{\mathcal{D}_{r,s,t}(\widehat{m}) \equiv \int_0^\infty \frac{1}{\left(\sqrt{\widehat{k}^2+\widehat{m}^2}\right)^r}\frac{\left(e^{\sqrt{\widehat{k}^2+\widehat{m}^2}}\right)^{s-1}}{\left(e^{\sqrt{\widehat{k}^2+\widehat{m}^2}}-1\right)^s}\widehat{k}^t d\widehat{k} \;\; \geq \;\; 0} \tag{20}$$

High-temperature limit further relies on $\beta m_0 \gg \frac{1}{768\pi}$ and $\lambda_R \ll \beta m_0$.

Summary

- Mass correction in $\lambda\varphi^4$-theory in expanding background: Coupled system of equations

- Description through WKB, valid for $\bar{\omega} \gg 4H$ or $-\bar{\omega} \ll 4H \Rightarrow$ connection formulae

- Quantitative understanding of WKB I: $m^2(t) \approx -\mu_R^2 + \left(\mu_R^2 + m^2(0)\right)\left(\frac{a(0)}{a(t)}\right)^2$

- Qualitative understanding of WKB II&III: low momentum modes grow exponentially, with $\int_{\mathrm{TP1}}^{\mathrm{TP2}} |\omega_k(t)|\, dt$.

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