Common Trends in Cosmology

and Particle Physics

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COSMOLOGICAL INFLATION

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Outline:

- Basics of Inflation
- Particle creation in classical backgrounds
  - General Theory
  - Examples
  - Applications to Cosmology
    - ⋆ Creation of Matter
    - ⋆ Generation of seeds for Structure
- Reheating after Inflation
  - Preheating
  - Turbulence
  - Thermalization

BASICS OF INFLATION

Puzzles of classical cosmology which Inflation solves:

WHY THE UNIVERSE

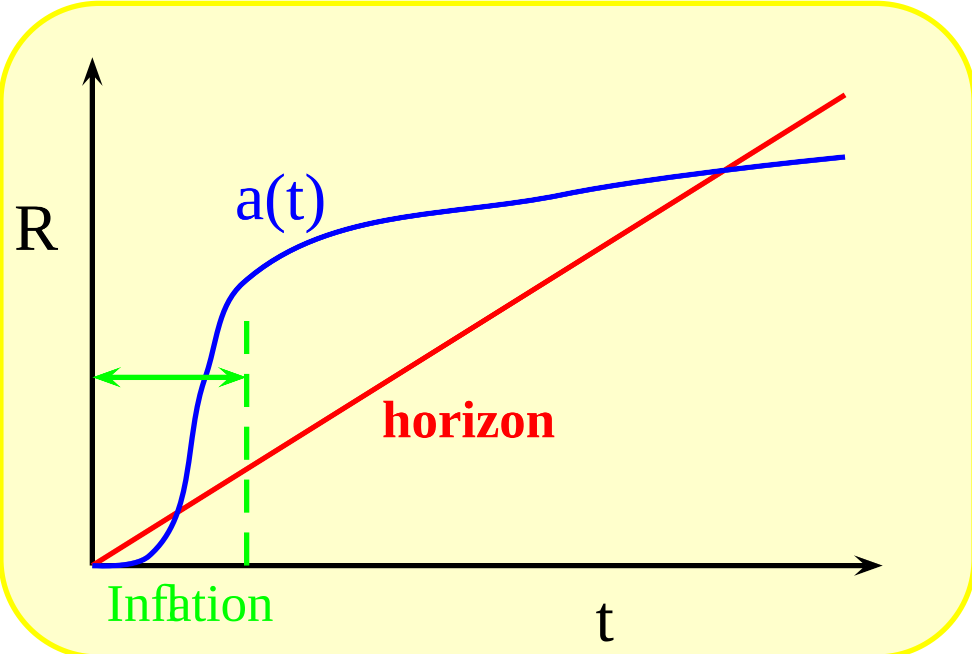
- is so old, big and flat ?
  $t > 10^{10}$ years
- homogeneous and isotropic?
  $\delta T/T \sim 10^{-5}$
- contains so much entropy?
  $S > 10^{90}$
- does not contain unwanted relics?
  (e.g. magnetic monopoles)

Horizon problem and the solution

Horison $\propto t$

Physical size $\propto a(t) \propto t^{\gamma}$

"Normal" Friedmann Universe: $\gamma < 1$



Inflationary Universe: $\gamma > 1$ or $\ddot{a} > 0$

$$\ddot{a} = -\frac{4\pi}{3} G a(\rho + 3p)$$

We have inflation when

$$p < -\rho/3$$

Getting something for nothing

$$T_\mu^{\ \nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

Energy-momentum conseravtion $T^{\mu\nu}{}_{;\nu} = 0$ can be written as

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0$$

Consider stress-energy tensor $T_{\mu\nu}$ for a vacuum. Vacuum has to be Lorentz invariant, hence $T_\mu^{\ \nu} = V\,\delta_\mu^{\ \nu}$ and we find $p = -\rho$

Energy of the vacuum stays constant despite the expansion !

Consider $T_{\mu\nu}$ for a scalar field φ

$$T_{\mu\nu} = \partial_\mu \varphi \, \partial_\nu \varphi - g_{\mu\nu} \, \mathcal{L}$$

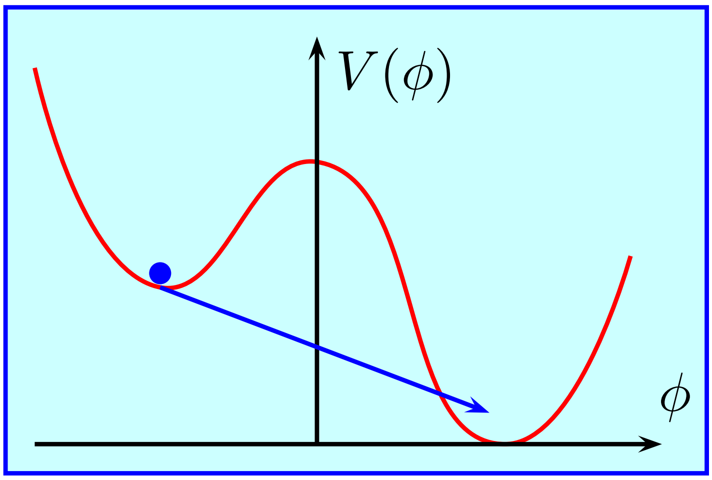
with the Lagrangian :

$$\mathcal{L} = \partial_\mu \varphi \, \partial^\mu \varphi - V(\varphi)$$

In a state when all derivatives of φ are zero, the stress-senergy tensor of a scalar field is that of a vacuum, $T_{\mu\nu} = V(\varphi) \, g_{\mu\nu}$.

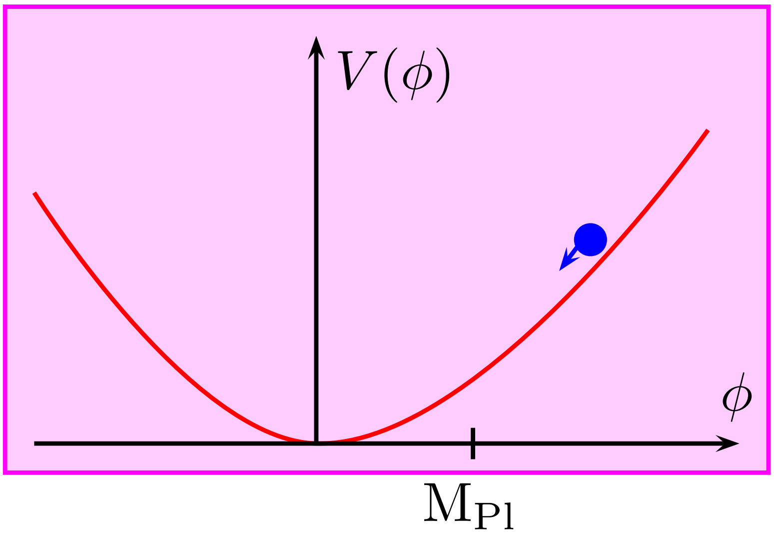
There are two basic ways to arrange $\varphi \approx \mathrm{const}$ and hence to imitate the vacuum-like state.

1. A. Guth: consider potential with two minima



2. A. Linde: consider the simplest potential

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2$$



Equation of motion

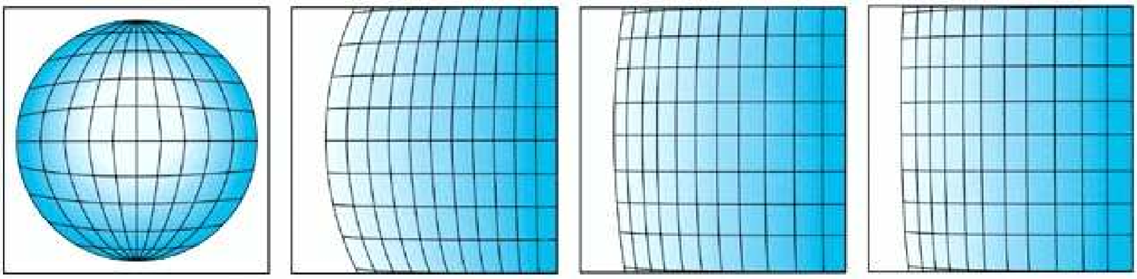
$$\ddot{\varphi} + 3H\dot{\varphi} + m^2\varphi = 0$$

If $\mathbf{H \gg m}$ the field (almost) does not move

$\mathbf{H \approx m} \frac{\varphi}{\mathbf{M_{Pl}}}$:

$\phi > \mathbf{M_{Pl}}$	Inflation
$\phi \sim \mathbf{M_{Pl}}$	End of Inflation
$\phi < \mathbf{M_{Pl}}$	Field oscillates.
	Reheating

Volume increases while the energy density stays constant.



Clean $(n \propto a^{-3})$ room for matter is created.

Crutual prediction: flat Universe, $\Omega = 1$.

But the Universe is in vacuum state.

Where all matter and seeds for structure formation came from ?

Unified theory of creation

Small fluctuations obey

$$\ddot{U}_k \; + \; [k^2 + m_{\rm eff}^2(\tau)] \, U_k = 0$$

It is not possible to keep fluctuations in vacuum if $m_{\rm eff}$ is time dependent.

Technical remarks:

- This is true for all species
- Equations look that simple in conformal refernce frame $ds^2 = a(\tau)^2 \; (d\tau^2 - dx^2)$
- For conformally coupled, but massive scalar $m_{\rm eff} = m_0 \, a(\tau)$
- $m_{\rm eff}$ may be non-zero even for massless fields.
  - graviton is the simplest example $m_{\rm eff}^2 = -\ddot{a}/a$
- Of particular interest are ripples of space-time itself
  - curvature fluctuations (scalar)
  - gravitons (tensor)

QFT in time-dependent background

Outline:

- General Theory
  - Bosons
  - Fermions
- Some analytical solutions
  - Parametric resonance
  - Parabolic cylinder functions
    - ⋆ Gravitational particle creation
    - ⋆ Stochastic resonance
- Transition to classical regime

General set-up

- Metric $ds^2 = a(\eta)^2(d\eta^2 - d\mathbf{x}^2)$
- Inflaton Lagrangian $L = \frac{1}{2}(\partial_\mu \varphi)^2 - V(\varphi)$
- Other fields (may interact with inflaton)
  - Scalar X:

$$V = \frac{1}{2}(m_X^2 - \xi R)X^2 + \frac{g^2}{2}\varphi^2 X^2$$

  - Fermion ψ:

$$V = (m_\psi + g\varphi)\,\bar{\psi}\psi$$

It is convenient to rescale fields, $\phi \equiv \varphi\, a(\eta)$ and $\chi \equiv X\, a(\eta)^s$, where $s = 1$ and $s = 3/2$ for scalar and fermion respectively. Fields are Fourier expanded.

The mode functions, e.g. of a scalar field are solutions of the oscillator equation

$$\ddot{g}_k + \omega_k^2\, g_k = 0\ ,$$

with the time dependent frequency

$$\omega_k^2 = k^2 - \frac{\ddot{a}}{a}(1 - 6\xi) + m_{\rm eff}^2(\phi)\, a^2$$

QFT in time-dependent background

Canonical Quantization

Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

Hamiltonian

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

Conjugated momenta

$$\pi(\mathbf{x}, t) = \frac{\delta\mathcal{L}}{\delta\dot{\phi}(\mathbf{x}, t)} = \dot{\phi}(\mathbf{x}, t)$$

Quantization

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y})\,. \tag{1}$$

Fourier transform

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3}\int d^3k\phi_{\mathbf{k}}(t)\mathrm{e}^{i\mathbf{kx}}$$

reduces equations of motion to

$$\ddot{\phi}_{\mathbf{k}} + \omega_k^2\phi_{\mathbf{k}} = 0\,,$$

where

$$\omega_k^2 = \mathbf{k}^2 + m^2\,.$$

Constraint $\phi_{\mathbf{k}} = \phi^*_{-\mathbf{k}}$ can be solved explicitly by

$$\phi_{\mathbf{k}}(t) \equiv \frac{(2\pi)^{3/2}}{\sqrt{2\omega_k}} \left(a_{\mathbf{k}}(t) + a^{\dagger}_{-\mathbf{k}}(t) \right) . \qquad (2)$$

Now we want to substitute the pair $\{\phi, \pi\}$ by the pair $\{a, a^{\dagger}\}$. Decomposition for π which complements (2) is

$$\pi(\mathbf{x}, t) = i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} (a^{\dagger}_{-\mathbf{k}} - a_{\mathbf{k}}) \mathrm{e}^{i\mathbf{k}\mathbf{x}} , \qquad (3)$$

and canonical commuitation relations (1) will be satisfied if

$$\left[a_{\mathbf{k}}(t), a^{\dagger}_{\mathbf{p}}(t) \right] = \delta(\mathbf{k} - \mathbf{p}) .$$

The Hamiltonian in terms of the $a_{\mathbf{k}}$ and $a^{\dagger}_{\mathbf{k}}$ operators can be written as $H \equiv H_{\mathrm{part}} + H_{\mathrm{vac}}(t)$, where

$$H_{\mathrm{part}} \equiv \int d^3k \, \omega_k a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} ,$$

$$H_{\mathrm{vac}}(t) \equiv \frac{V}{(2\pi)^3} \int d^3k \, \frac{\omega_k}{2}$$

This procedure goes through even if ω is time dependent.

The Fock space

Let us introduce the vacuum state $|0_t\rangle$

$$a_{\mathbf{k}}(t)|0_t\rangle = 0\,.$$

Here t is some specified (but arbitrary at this point) moment of time. The state

$$|n_k\rangle = (a^\dagger_{\mathbf{k}})^{n_k}|0_t\rangle$$

can be interpreted as a state which contains n_k particles, each with energy ω_k. Indeed

$$H_{\rm part}|n_k\rangle = \omega_k\, n_k\, |n_k\rangle\,.$$

and

$$N = \int d^3p\; a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$$

counts the number of particles, $N|n_k\rangle = n_k\,|n_k\rangle$.

In the vacuum state, $|0_t\rangle$, the energy takes its lowest possible value at this moment of time

$$H_{\rm vac}(t) \equiv \langle 0_t|H|0_t\rangle\,.$$

This procedure goes through even if ω is time dependent.

Equations of motion

$$\frac{da_{\mathbf{k}}}{dt} = \frac{\partial a_{\mathbf{k}}}{\partial t} + i[H, a_{\mathbf{k}}]$$

Let us invert relations (2) and (3)

$$a_{\mathbf{k}} = \frac{1}{\sqrt{2}} \int \frac{d^3x}{(2\pi)^{3/2}} \, \mathrm{e}^{-i\mathbf{k}\mathbf{x}} \left(\sqrt{\omega_k}\phi + i\frac{\pi}{\sqrt{\omega_k}} \right) ,$$

$$a^{\dagger}_{-\mathbf{k}} = \frac{1}{\sqrt{2}} \int \frac{d^3x}{(2\pi)^{3/2}} \, \mathrm{e}^{-i\mathbf{k}\mathbf{x}} \left(\sqrt{\omega_k}\phi - i\frac{\pi}{\sqrt{\omega_k}} \right) .$$

The original canonical variables $\{\phi, \pi\}$ do not have explicit time dependence, $\partial\phi/\partial t = \partial\pi/\partial t = 0$,

but the ω_k can be time-dependent.

$$\frac{da_{\mathbf{k}}}{dt} = -i\omega_k a_{\mathbf{k}} + \frac{1}{2}\frac{\dot{\omega}_k}{\omega_k}\, a^{\dagger}_{-\mathbf{k}} \qquad (4)$$

We see that the solution of the equations of motion for operators can be parametrized as

$$\begin{aligned} a_{\mathbf{k}}(t) &= \alpha_k(t)\, a_{\mathbf{k}}(0) + \beta_k(t)\, a^{\dagger}_{-\mathbf{k}}(0) \\ a^{\dagger}_{\mathbf{k}}(t) &= \alpha^*_k(t)\, a^{\dagger}_{\mathbf{k}}(0) + \beta^*_k(t)\, a_{-\mathbf{k}}(0) \end{aligned} \qquad (5)$$

The commutation relations should be satisfied at any moment of time, therefore α and β obey the constraint

$$|\alpha_k|^2 - |\beta_k|^2 = 1\,.$$

An immediate consequence of the relations (5) is that the system which was in vacuum initially, $a_{\mathbf{k}}(0)|0\rangle = 0$ will not remain in vacuum as the time goes by

$$a_{\mathbf{k}}(t)|0\rangle = \beta_k(t) a^\dagger_{-\mathbf{k}}(0)|0\rangle \neq 0$$

In particular, the number density of particles created from the vacuum is

$$n(t) = \frac{1}{V}\langle 0|N|0\rangle = \frac{1}{(2\pi)^3}\int d^3k\; |\beta_k(t)|^2$$

To find $\beta_k(t)$ we substitute Eq. (5) into Eq. (4)

$$\begin{aligned}\dot{\alpha}_k &= -i\omega_k\alpha_k + \frac{1}{2}\frac{\dot{\omega}_k}{\omega_k}\,\beta^*_k \\ \dot{\beta}_k &= -i\omega_k\beta_k + \frac{1}{2}\frac{\dot{\omega}_k}{\omega_k}\,\alpha^*_k\end{aligned}$$

Intial conditions are fixed $\alpha_k(0) = 1$ and $\beta_k(0) = 0$. With $\omega_k(t)$ being given we solve this system of four ordinary differential equations and

This is it for the general theory !

Adiabaticity condition

The number of particles created during the time interval $\Delta t \sim \omega_k^{-1}$ is

$$|\,\Delta|\beta_k|^2\,| < \frac{1}{4}\left(\frac{\dot{\omega}_k}{\omega_k^2}\right)^2 .$$

The particle number is conserved approximately if

$$\left|\frac{\dot{\omega}_k}{\omega_k^2}\right| \ll 1\,.$$

Such approximately conserved quantities are called adiabatic invariants.

Mode Functions

One can do field decomposition over time independent operators as well

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(g_k(t)\, a_{\mathbf{k}}(0)\, \mathrm{e}^{i\mathbf{kx}} + \mathrm{h.c.} \right) \; .$$

Equation of motion for the mode functions

$$\ddot{g}_k + \omega_k^2 g_k = 0 \, .$$

Comparing to decomposition of $\phi(\mathbf{x}, t)$ over $a(t)$ we find immideately

$$\beta_k^* = \frac{\omega_k g_k - i\dot{g}_k}{\sqrt{2\omega_k}} \; ,$$

$$\alpha_k = \frac{\omega_k g_k + i\dot{g}_k}{\sqrt{2\omega_k}} \; .$$

This gives in particular

$$|\beta_k|^2 = \frac{|\dot{g}_k|^2 + \omega_k^2 |g_k|^2}{2\omega} - \frac{1}{2} \; .$$

Diagonalization of the Hamiltonian

$$H = \int d^3k \, [\, E_k(t) \left(a^\dagger_{\mathbf{k}}(0) a_{\mathbf{k}}(0) + a_{\mathbf{k}}(0) a^\dagger_{\mathbf{k}}(0) \right)$$

$$+ F_k(t) a_{\mathbf{k}}(0) a_{-\mathbf{k}}(0) + F^*_k(t) a^\dagger_{\mathbf{k}}(0) a_{-\mathbf{k}}(0)^\dagger \,] \, ,$$

where

$$E_k(t) = \frac{1}{2} \left(|\dot{g}_k|^2 + \omega_k^2 |g_k|^2 \right) \; ,$$

$$F_k(t) = \frac{1}{2} \left(\dot{g}_k^2 + \omega_k^2 g_k^2 \right) \; .$$

Bogolyubov's transformation:

$$a_{\mathbf{k}} = \alpha_k b_{\mathbf{k}} + \beta_k b^\dagger_{-\mathbf{k}} \, ,$$

$$a^\dagger_{\mathbf{k}} = \alpha^*_k b^\dagger_{\mathbf{k}} + \beta^*_k b_{-\mathbf{k}} \, .$$

$$|\beta_k|^2 = \frac{2E_k - \omega_k}{2\omega_k} \, .$$

Fermions

Heisenberg equations of motion give

$$\dot{\alpha}_k = -i\omega_k \alpha_k + \frac{k\dot{m}}{2\omega_k^2}\beta_k^* \ ,$$

$$\dot{\beta}_k = -i\omega_k \beta_k - \frac{k\dot{m}}{2\omega_k^2}\alpha_k^* \ .$$

In terms of mode functions:

$$\ddot{u}_\pm + (\omega_k^2 \pm i\dot{m})u_\pm = 0\,.$$

we have

$$|\beta_k|^2 = \frac{\omega_k \pm m + \mathrm{Im}(u_\pm^* \dot{u}_\pm)}{2\omega_k}\ .$$

Initial conditions: vacuum

If we are working in terms of α_k and β_k, the vacuum initial conditions correspond to

$$\alpha_k(0) = 1, \qquad \beta_k(0) = 0.$$

If we are working in terms of mode functions, the vacuum initial conditions can be obtained using already displayed relations between both sets of variables, e.g.

$$g_k(t) = \frac{\alpha_k + \beta_k^*}{\sqrt{2\omega_k}} .$$

$$\dot{g}_k(t) = i\sqrt{\frac{\omega_k}{2}}\,(\beta_k^* - \alpha_k)$$

We obtaien

- Bosons

$$g_k(0) = \frac{1}{\sqrt{2\omega}}, \qquad \dot{g}_k(0) = -i\omega g_k(0)$$

- Fermions

$$u_k(0) = \sqrt{1 - \frac{m_{\rm eff}}{\omega}}, \qquad \dot{u}_k(0) = -i\omega u_k(0)$$

Particle number vs variance

Useful quantities

- Particle number
  - $n_k = |\beta_k|^2$
  - Adiabatic invariant at sub-horizon scales (if $m > H$)
  - Allows to calculate e.g. dark matter abundances
  - But has no meaning at super-horizon scales
- Field variance
  - $\langle \phi^2 \rangle$
  - Does not evolve at super-horizon scales (if $m < H$)
  - Allows to calculate density perturbations generated during inflation
  - Crutial for dynamics of phase transitions
  - Helps to calculate back-reaction in a simple way (Hartree approximation)
  - But evolves on sub-horizon scales

Variances

Bose field

$$\langle 0|\phi^2(x)|0\rangle_{\text{reg}} = \int \frac{d^3k}{(2\pi)^3}\, \frac{|\beta_k|^2 + \text{Re}(\alpha_k\beta_k)}{\omega_k}$$

or

$$\langle 0|\phi^2(x)|0\rangle_{\text{reg}} = \int \frac{d^3k}{(2\pi)^3} \left(|g_k|^2 - \frac{1}{2\omega_k}\right)$$

Fermion field

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\text{reg}} = 2\sum_s \int \frac{d^3k}{(2\pi)^3}\, \frac{m\,|\beta_k|^2 - k\,\text{Re}(\alpha_k\beta_k)}{\omega_k}$$

or

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{\text{reg}} = 2\int \frac{d^3k}{(2\pi)^3} \left[|u_-|^2 + \frac{m}{\omega_k} - 1\right]$$

Parametric resonance

Consider system of two intercting fields

$$V_{\rm int}(\chi,\varphi) = \frac{1}{2}M^2\varphi^2 + \frac{1}{2}m^2\chi^2 + \frac{1}{2}g^2\varphi^2\chi^2 \; .$$

Assume the field φ has non-zero expectation value $\varphi(t) = \varphi_0 \cos(Mt)$. This creates effective mass for the field χ: $m^2_{\rm eff} = m^2 + g^2\varphi^2$ and

$$\omega_k^2(t) = \mathbf{k}^2 + m^2 + \frac{1}{2}g^2\varphi_0^2 + \frac{1}{2}g^2\varphi_0^2\cos(2Mt)$$

Equation for the mode functions

$$\ddot{g}_k + \omega_k^2 g_k = 0\,.$$

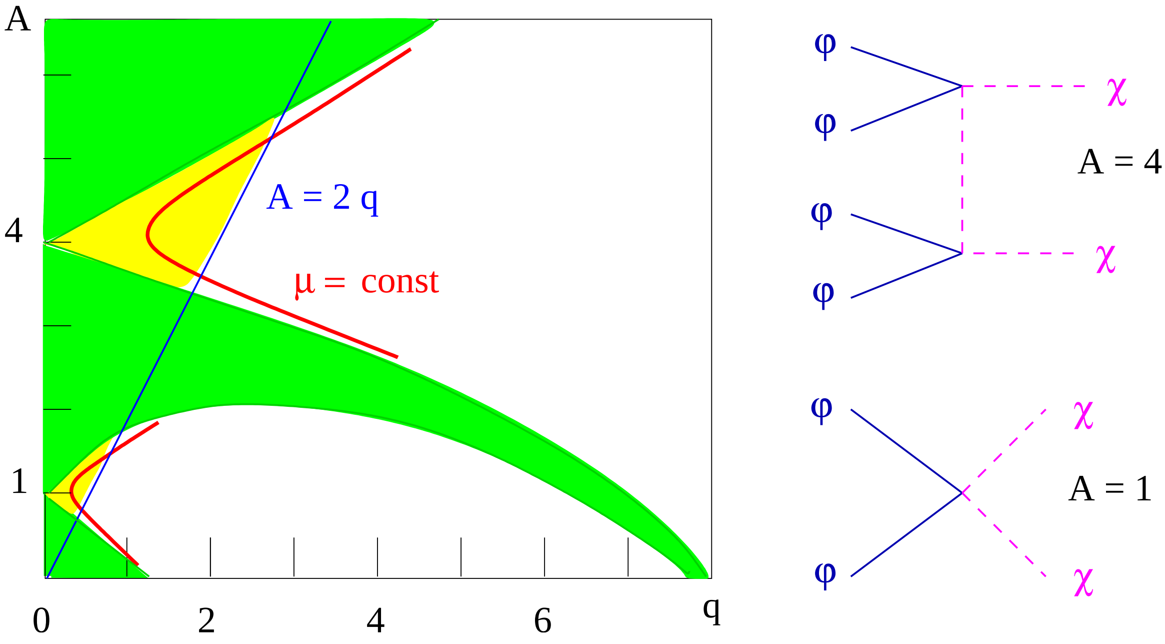
can be reduced to the standard form of the Mathieu equation

$$g''_{\mathbf{k}} + [A_k - 2q\cos 2\tau]g_{\mathbf{k}} = 0\,,$$

where $q \equiv \frac{g^2\varphi_0^2}{4M^2}$ and $A_k \equiv \frac{k^2+m^2}{M^2} + 2q$ $(A > 2q)$.

Stability-Intstability zones

$$q = \frac{g^2 \varphi_0^2}{4M^2} \qquad A_k = \frac{k^2 + m^2}{M^2} + 2q$$



In unstable bands (yellow) $g \propto e^{\mu\tau}$

$$\mu_1 = \frac{q\, m}{2}, \qquad \mu_2 = \frac{q^2\, m}{16}, \quad \ldots$$

$$\delta k \approx \mu$$

Parabolic Cylinder Functions

Analytical solutions of a large class of problems of particle creation in time varying background can be expressed in terms of the well studied parabolic cylinder functions. These are solutions of the equation

$$\frac{d^2y}{d\tau^2} + (\frac{1}{4}\tau^2 + \nu)\, y = 0 \tag{6}$$

Particle creation during "short" non-adiabatic intervals

Assume $\omega(t)$ goes through a minimum:

$$\omega_k^2(t) = \omega_k^2(t_*) + \frac{1}{2}\omega_k^2{}''(t_*)(t - t_*)^2 + \ldots$$

and change time variable to $\tau \equiv \left[2\omega_k^2{}''(t_*)\right]^{\frac{1}{4}}\,(t - t_*)$.
Equation for mode functions reduces to Eq. (6) with

$$\nu_k \equiv \frac{\omega_k^2(t_*)}{\sqrt{2\omega_k^2{}''(t_*)}}\,.$$

The answer:

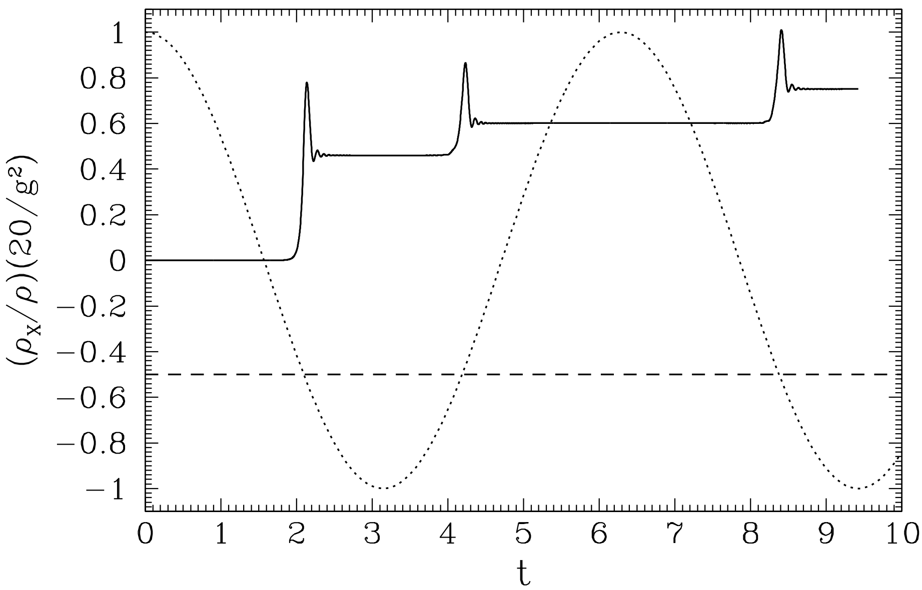
$$|\beta_k|^2 = \mathrm{e}^{-2\pi\nu_k}$$

1. Coupling to classical scalar field

Consider Fermion ψ coupled to classical scalar field $\phi(t)$, $\mathcal{L}_Y = g\phi\bar{\psi}\psi$. The effective mass of the fermion field

$$m_{\rm eff}(t) = m_\psi + g\phi(t)\,.$$

Creation occurs at $\quad m_{\rm eff} = 0:$



We can disregard details of evolution and write

$$m_{\rm eff} = g{\phi_*}'\,(t - t_0)$$

Equation for mode function reduces to

$$u'' + (p^2 - i + \tau^2)u = 0$$

where $p \equiv k/\sqrt{g\phi'_*}$ and $\tau \equiv (t - t_0)\sqrt{g\phi'_*}$

EXAMPLES

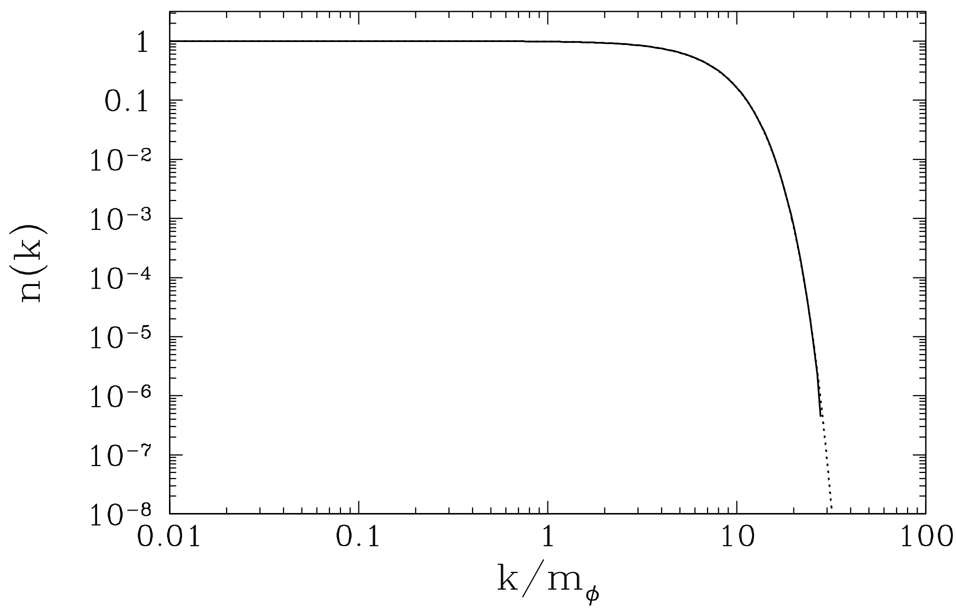
Solutions are Parabolic Cylinder functions and

$$n(k) = \exp\left(-\pi k^2/g\phi'_*\right)$$

For harmonic oscillations in flat space-time this gives

$$n(k) = \exp\left(\frac{-\pi k^2}{m_\phi^2\sqrt{4q - m_X^2/m_\phi^2}}\right)$$

$$q \equiv \frac{g^2\phi^2(0)}{4m_\phi^2}$$



Solid line: numerical integration of complete problem with

$$q = 10^4 \quad \text{and} \quad m_X/m_\phi = 100 \; .$$

Dotted line: analytical approximation based on Parabolic Cylinder functions.

2. Gravitational particle production in an expanding Friedmann Universe

In conformal reference frame

$$ds^2 = a^2(t)(d\tau^2 - d\mathbf{x}^2)$$

the frequency of field $\chi \equiv a\phi$ is

$$\omega_k^2 = k^2 + m^2 a^2 - \frac{a''}{a}(1 - 6\xi)\,,$$

where ξ is coupling to curvature, $\frac{1}{2}\xi R\phi$.

In radiation dominated universe $a'' = 0$ and $a(\tau) = H_0\tau$.

Problem is reduced to Parabolic Cylinder Functions with

$$\nu_k \equiv \frac{k^2}{2mH_0}\,.$$

This gives

$$n = 1.495 \times 10^{-3}\,\frac{(mH_0)^{\frac{3}{2}}}{a^3}$$

Regime is adiabatic at $\tau > \tau_* = 1/\sqrt{mH_0}$.

Therefore, particles are created when $H > m$. In general,

$$n = \frac{m^3}{a^3}C$$

Schroedinger picture of evolution

Find $U(t)$ such that

$$a_k(t) = U^\dagger(t) a_k(0) U(t) \,.$$

Solution of the Schroedinger equations of motion

$$|\psi(t)\rangle = U(t)|0\rangle$$

($|\psi(t)\rangle$ is called Squeezed state).
Clearly, vacuum at time t is given by

$$|0_t\rangle = U^\dagger(t)|0\rangle \,.$$

Since we know $a(t)$, we can also find

$$U a_k(0) U^\dagger = \alpha_k^* \, a_{\mathbf{k}}(0) - \beta_k a^\dagger_{-\mathbf{k}}(0) \,.$$

This product annihilates $|\psi(t)\rangle$, i.e. Schroedinger equation can be written as

$$U a_k(0) U^\dagger \; |\psi(t)\rangle = 0$$

Expressing a_k via field and its conjugate momenta gives

$$U a_k(0) U^\dagger = \frac{(\alpha_{\mathbf{k}}^* - \beta_{\mathbf{k}})\omega_k \phi_{\mathbf{k}} + i(\alpha_k^* + \beta_k)\pi_{\mathbf{k}}}{\sqrt{2\omega_k}}$$

($\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$ should be taken at the initial moment of time.)

Quantum to classical transition

Therefore, $|\psi(t)\rangle$ satisfies Schroedinger equation

$$(\Omega_k \phi_{\mathbf{k}} + i\pi_{\mathbf{k}})\,|\psi(t)\rangle = 0\,,$$

where

$$\Omega_k \equiv \frac{\alpha_k^* - \beta_k}{\alpha_k^* + \beta_k}\omega_k\,,$$

and

$$\pi_{\mathbf{k}} = -i\frac{\partial}{\partial\phi_{-\mathbf{k}}}\,.$$

This equation is easy to solve

$$\psi(\phi_{\mathbf{k}}, t) = \mathrm{e}^{-\Omega_k \phi_{-\mathbf{k}}\phi_{\mathbf{k}}} = \mathrm{e}^{-\Omega_k |\phi_{\mathbf{k}}|^2}\,.$$

In particular, this gives for the probability distribution of field values

$$P(\phi_{\mathbf{k}}, t) = |\psi(\phi_{\mathbf{k}}, t)|^2 = \mathrm{e}^{-|\phi_{\mathbf{k}}|^2/|g_k|^2}\,.$$

Cosmological Applications

Outline:

- Gravitational particle creation
- Coupling to the inflaton as a source of creation
  - Efficiency of particle creation as function of coupling and mass. Hartree approximation
  - Comparison of Fermi and Bose cases
- Lattice results.
  - Efficiency of particle creation
  - Non-thermal phase transitions
- Particle creation during inflation
  - Generation of density perturbations
  - Probe of trans-Plankian physics ?

Sources of creation

- Expansion of space-time itself, $a(\tau)$
- Motion of the inflaton field, $\phi(\tau)$

Both can be operational at any

Epoch of creation

- During inflation
  (superhorizon size perturbations)
- While the inflaton oscillates
  (reheating)

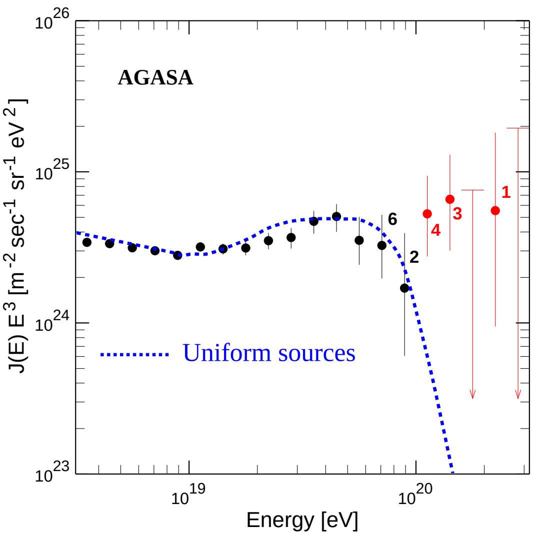
Gravitational creation of matter

$$m_{\rm eff} = m_0\, a(\tau)$$

$$\Downarrow$$

Superheavy Dark Matter

(WIMPzilla)



Ultra High Energy Cosmic Rays ?

Matter dominated Universe, $\rho = mn$.

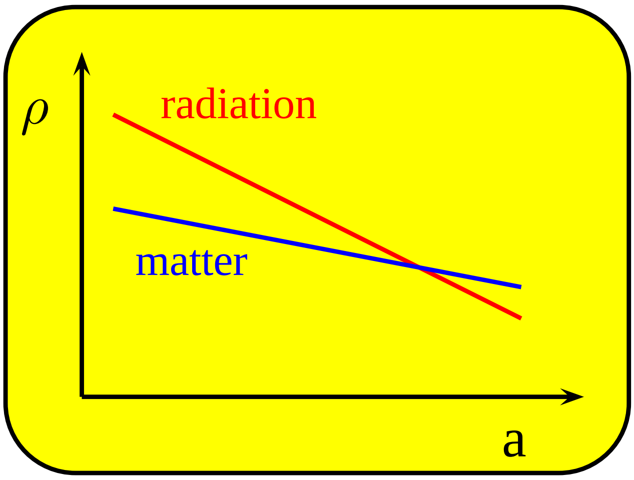
Baryon number conservation $(N = na^3 = \text{const})$:

$$\rho \propto a^{-3} \qquad a \propto t^{-2/3}$$

Radiation dominated Universe, $\rho = T^4$.

Entropy conservation $(S = T^3 a^3 = \text{const})$:

$$\rho \propto a^{-4} \qquad a \propto t^{-1/2}$$



$$T_{\text{eq}} \sim 1 \text{ eV}$$

Even tiny initial amount of matter
may show up at present

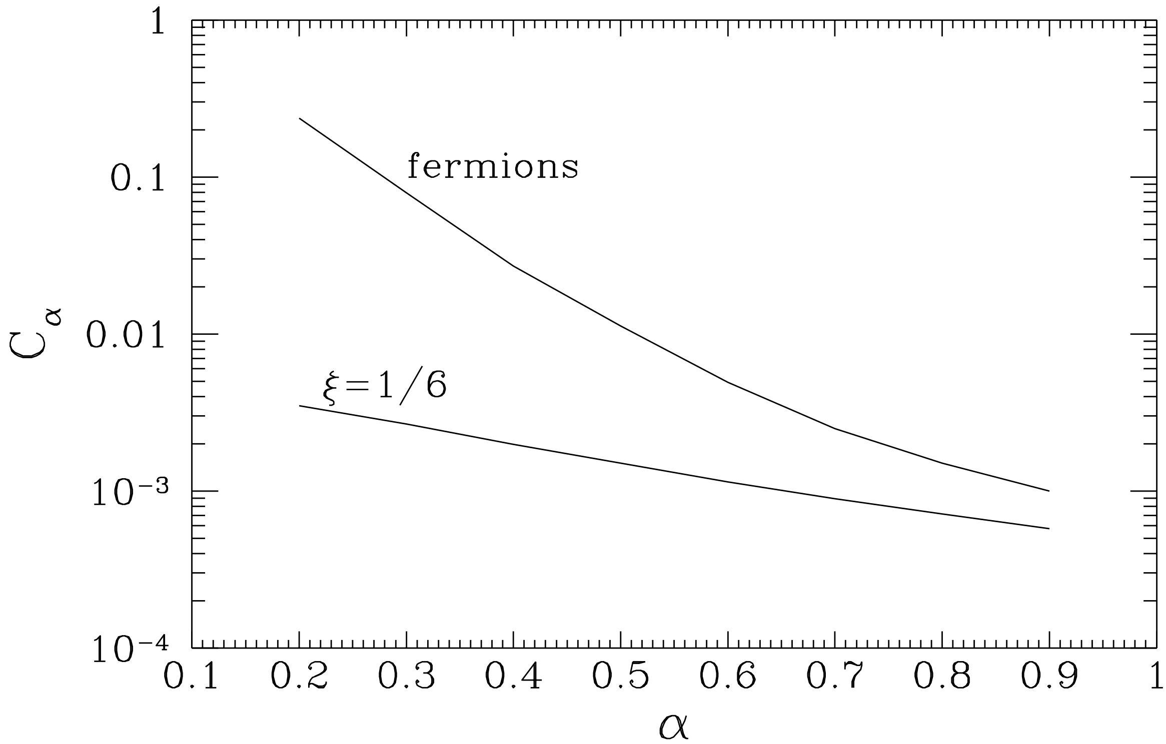
FRIEDMANN COSMOLOGY

It is the particle mass which couples the system to the background expansion and serves as the source of particle creation. Therefore we expect

$$n_X \propto m_X^3 a^{-3}$$

In Friedmann cosmology, $a \propto (mt)^\alpha \propto (m/H)^\alpha$,

$$n_X = C_\alpha m_X^3 \left(\frac{H}{m_X}\right)^{3\alpha}$$



Stable particles with $m_X > 10^9$ GeV will overclose the Universe.

Kuzmin & I.T. (1998)

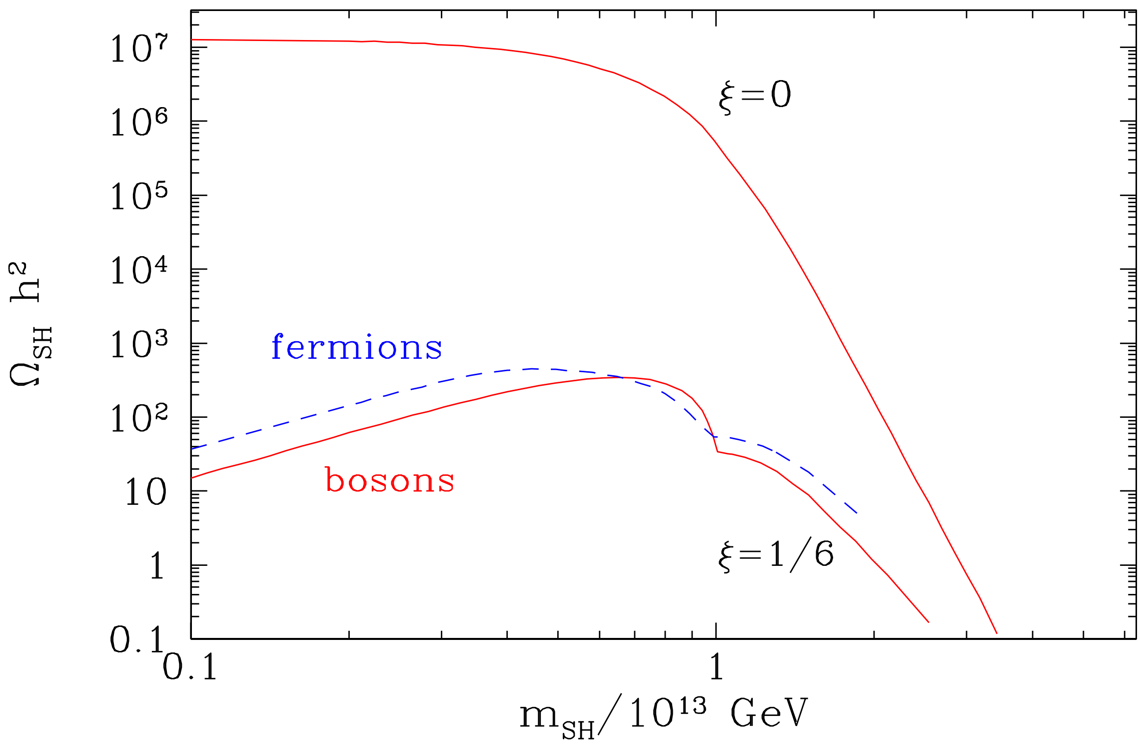
INFLATIONARY COSMOLOGY

There is no singularity and Hubble constant is limited,

$$H < m_\phi$$

Production of particles with $m_X > H \sim 10^{13}$ GeV is suppressed.

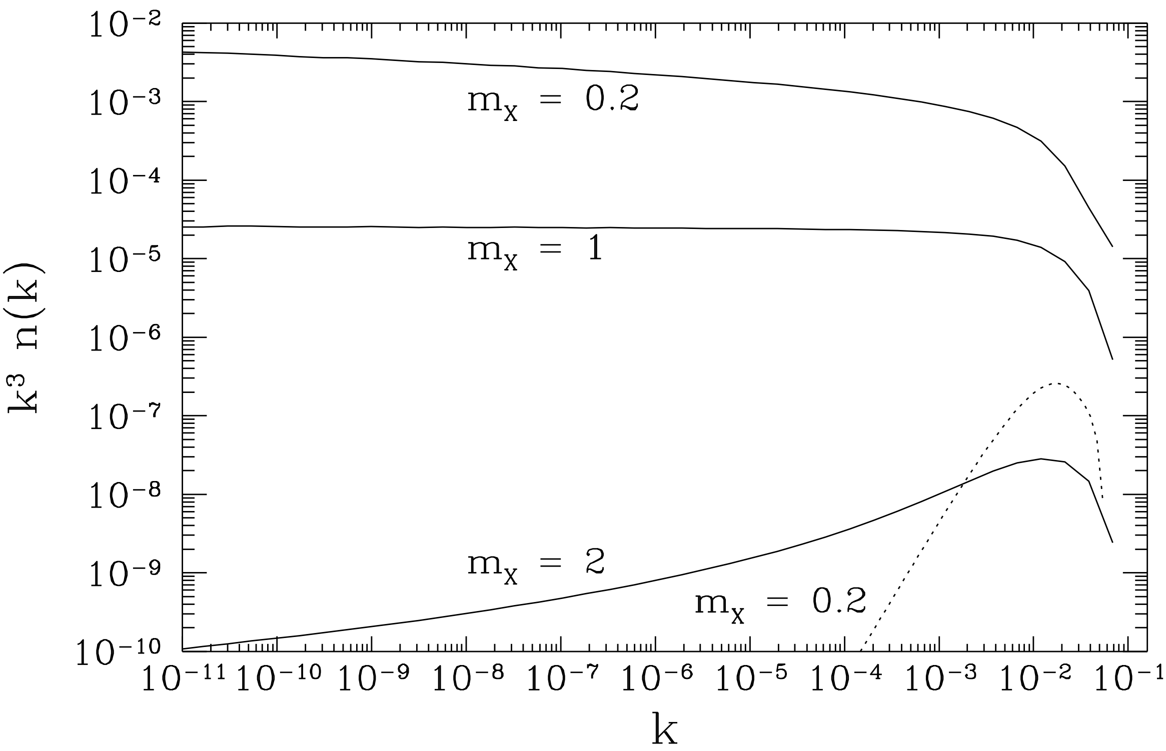
Present day ratio of the energy density in X-particles to the critical energy density:



Kuzmin & I.T. (1998)

Chung, Kolb & Riotto (1998)

Spectrum of density perturbations



Dark matter density fluctuations induced in the process of X particle creation can contrubute to fluctuations in CMBR at horizon scale if $m_X < 3$.

Kuzmin & I.T. (1998)

Coupling to the inflaton as a source of creation

scalar X	$m_{\text{eff}}^2 = m_X^2 + g^2\phi^2(t)$	$L_{\text{int}} = \frac{1}{2}g^2\phi^2X^2$
fermion ψ	$m_{\text{eff}} = m_\psi + g\phi(t)$	$L_{\text{int}} = g\phi\bar{\psi}\psi$

Numerology

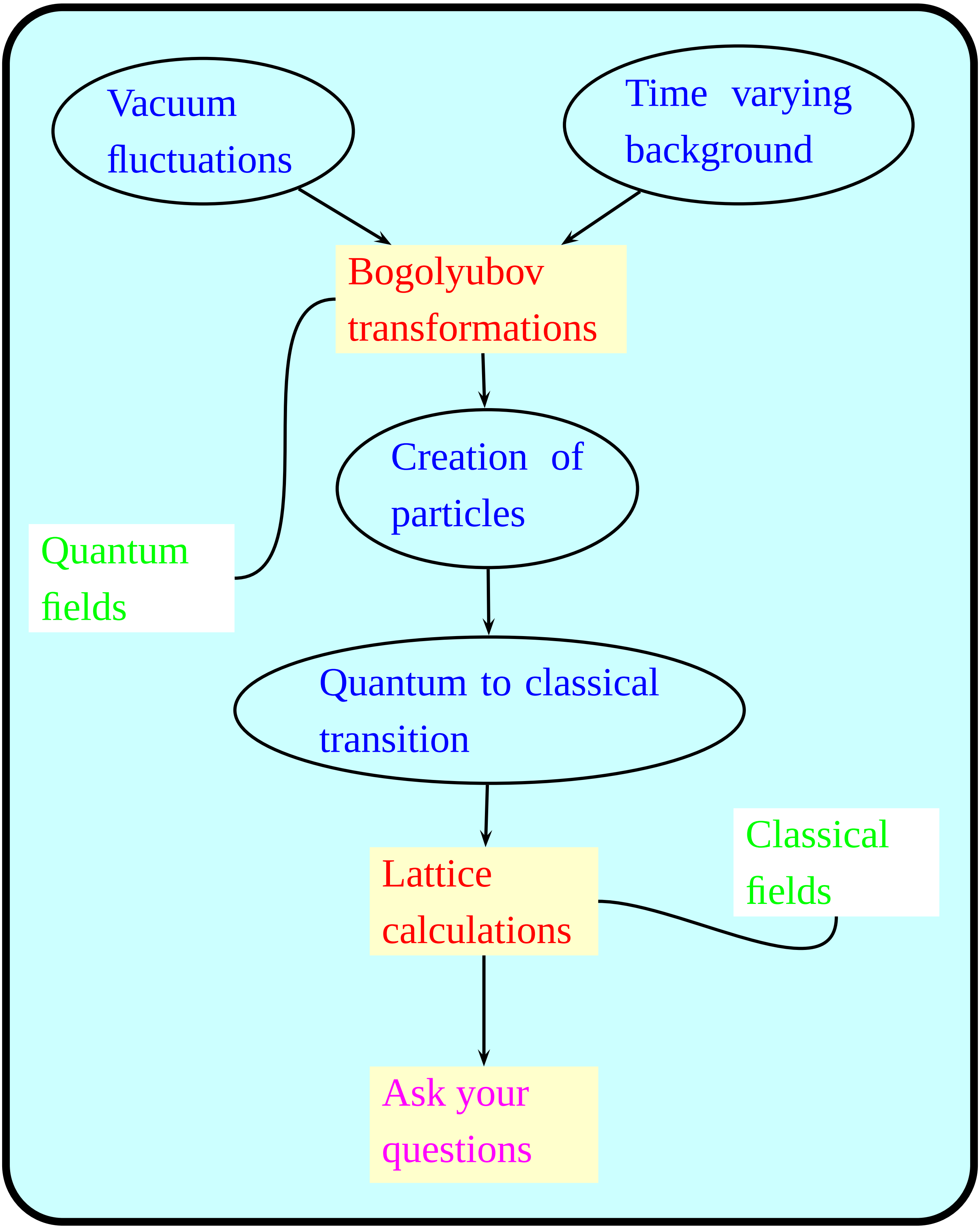
Rescaled coupling:

$$g^2 \to q \equiv \frac{g^2\phi^2}{4m_\phi^2}$$

$$\frac{\phi^2}{m_\phi^2} \approx \left(\frac{10^{19}\ \text{GeV}}{10^{13}\ \text{GeV}}\right)^2 \approx 10^{12}$$

and q can be enormous.

Methodology



Bose versus Fermi :

Bose stimulation.
Occupation numbers
grow, $n = e^{\mu t}$

↓

Fast, explosive decay
of the inflaton
(1994)
Kofman, Linde, Starobinsky;
Traschen, Brandenberger

↓

Large classical fluctuations
$\Downarrow$
non-thermal phase transitions
Kofman, Linde & Starobinsky (96);
I. T. (96)

Pauli blocking.
Occupation numbers
$n < 1$

Bose versus Fermi :

Effective mass

$m_{\rm eff}^2 = m^2 + g^2\phi^2$

Heavy particles are always heavy

Effective mass

$m_{\rm eff} = m + g\phi$

Heavy particles are massless at $\phi = -\frac{m}{g}$

→

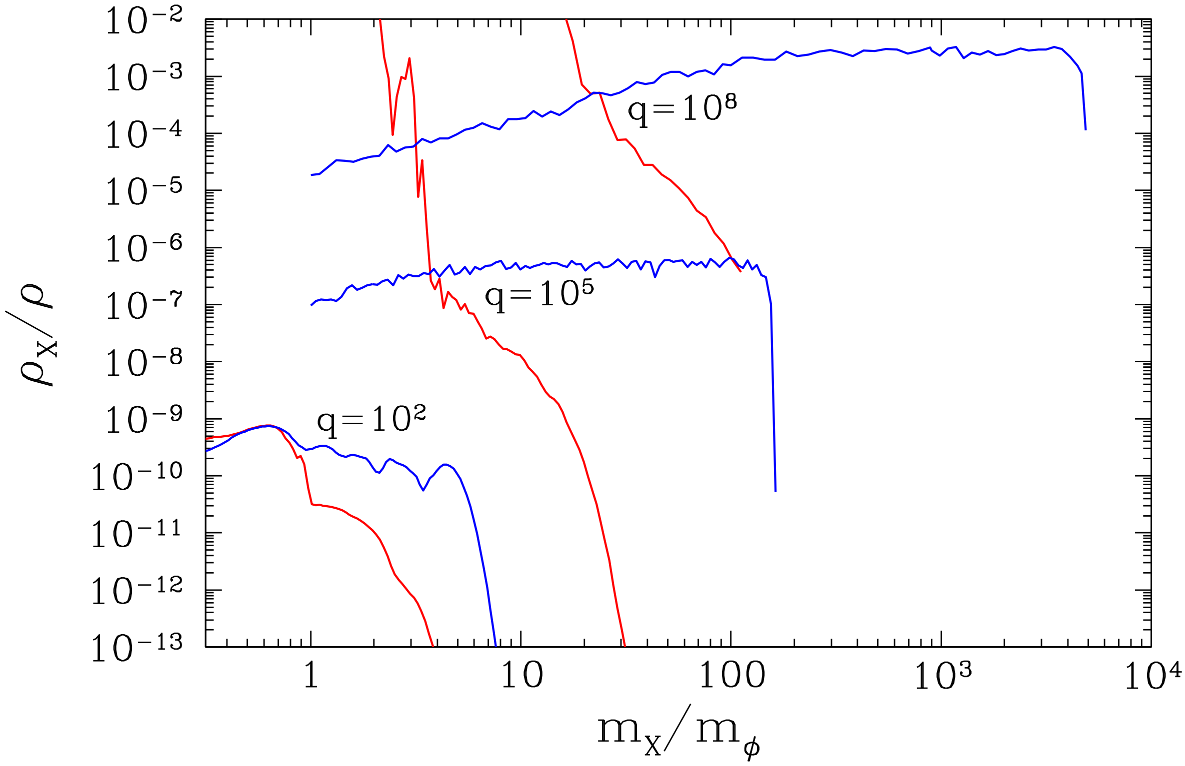
Superheavy fermion creation

Giudice, Peloso, Riotto, & I. T. (99)

Matter creation: Bose versus Fermi

Effectiveness of X-particles production in
$V(\phi) = m_\phi^2 \phi^2/2$ inflaton model

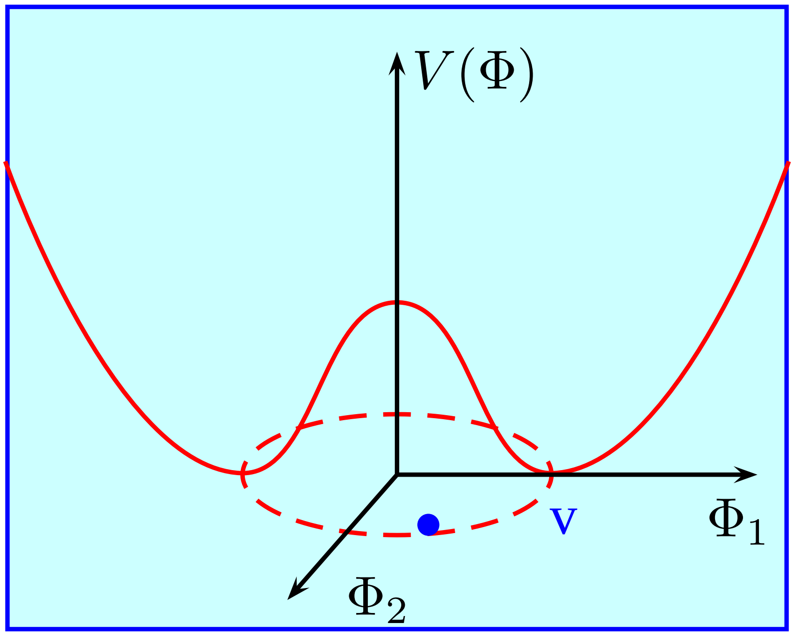
$$\mathbf{q} \equiv \frac{\mathbf{g^2 \phi^2(0)}}{\mathbf{4m_\phi^2}}$$



Blue lines: production of Fermions.

Red lines: production of Bosons.

Symmetry behaviour in a medium



$$V(\Phi) = -\mu^2\Phi^2 + \lambda\Phi^4 = \lambda(\Phi^2 - \mathrm{v}^2)^2$$

Consider coupling $g^2X^2\Phi^2$

Effective mass of Φ in a medium

$$m^2_{\text{eff}} = -\mu^2 + g^2\langle X^2\rangle$$

Symmetry is restored if $\langle X^2\rangle > \frac{\mu^2}{g^2}$

Large Variances at Preheating

Consider inflaton $\rightarrow$ X

Energy density in X :

$$\rho_X \propto \dot{X}^2 + \nabla X^2 \approx E^2 \langle X^2 \rangle$$

Assume the decay is instantaneous

$$\rho_X \sim m^2 M_{\rm Pl}^2$$

We find

$$\langle X^2 \rangle \sim \frac{\rho}{E^2} \sim \frac{m^2 M_{\rm Pl}^2}{m^2} \sim M_{\rm Pl}^2$$

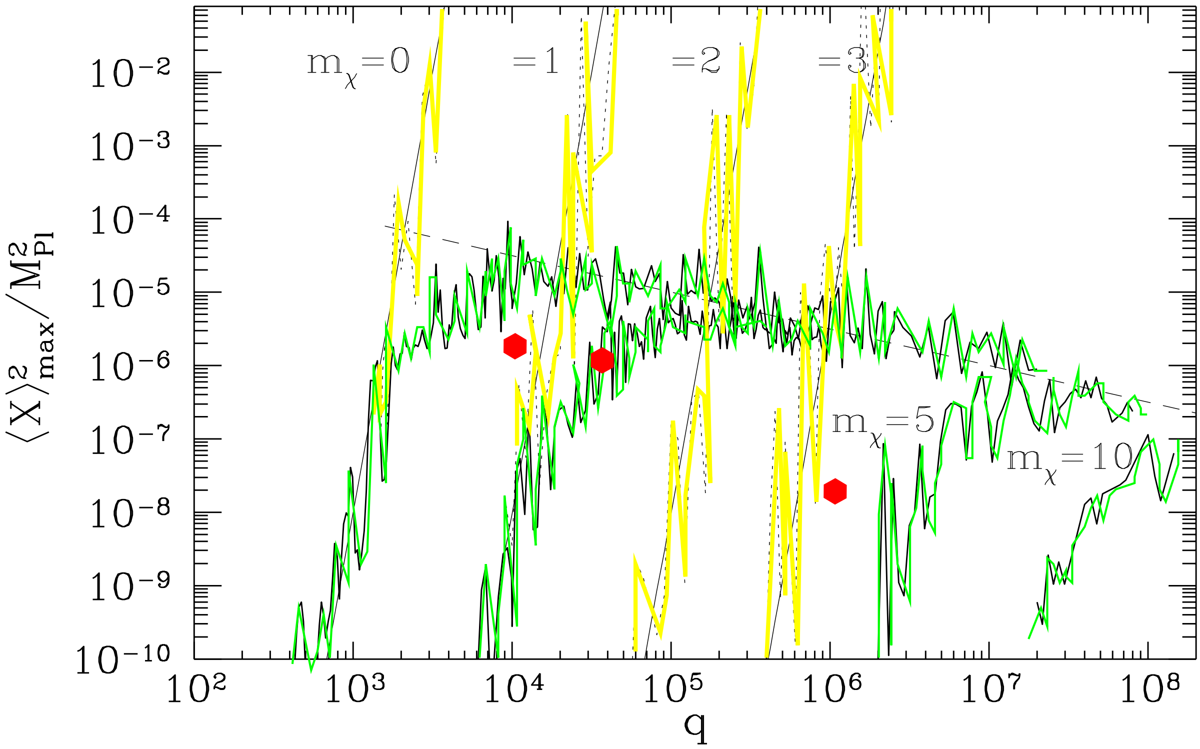
In thermal equilibrium

$$\langle X^2 \rangle = \frac{T^2}{12} \ll M_{\rm Pl}^2$$

Kofman, Linde, Starobinsky (1996)
IT (1996)

Field variances (Bosons)

$$V = \frac{m^2}{2}\phi^2 + \frac{g^2}{2}\phi^2 X^2 + \frac{m_X^2}{2} X^2$$



No back reaction

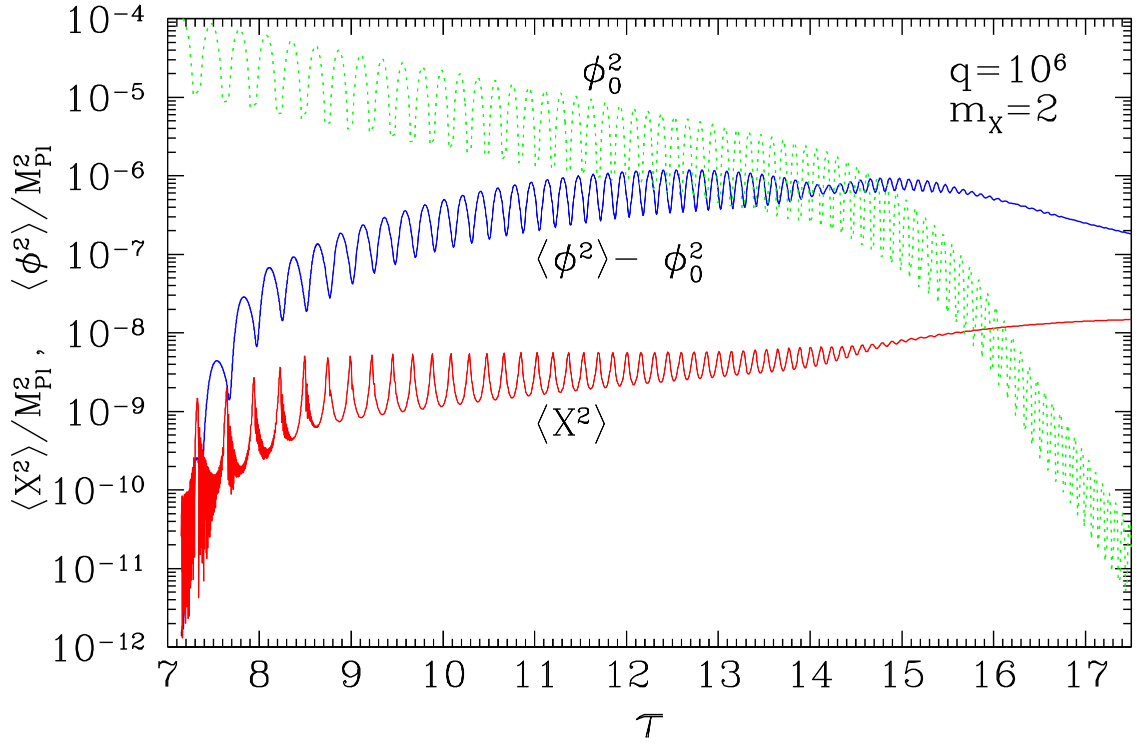
Hartree approximation

Exact lattice result , massless χ

$$q \equiv \frac{g\phi_0^2}{4m^2}$$

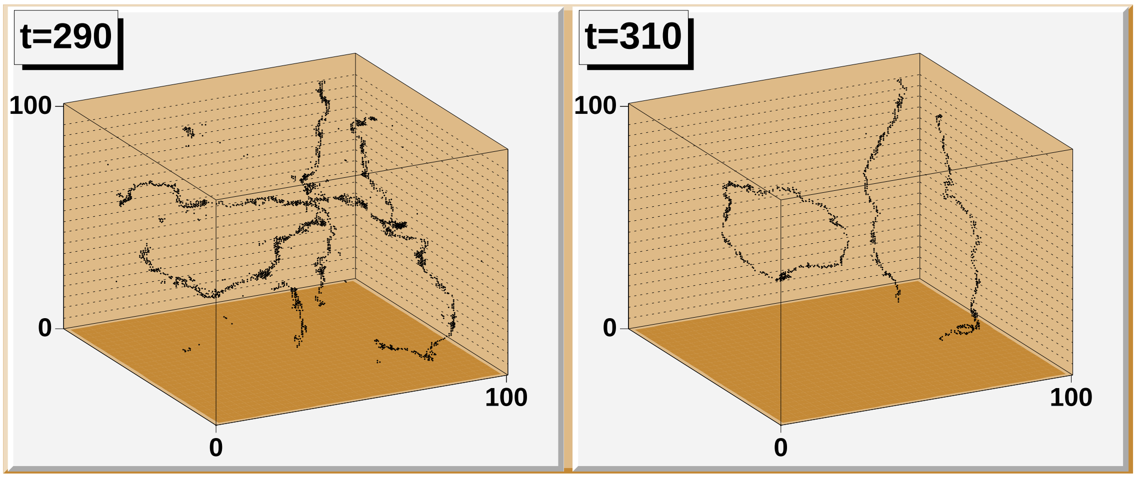
Field variances (Bosons)

$$V = \frac{m^2}{2}\phi^2 + \frac{g^2}{2}\phi^2 X^2 + \frac{m_X^2}{2} X^2$$

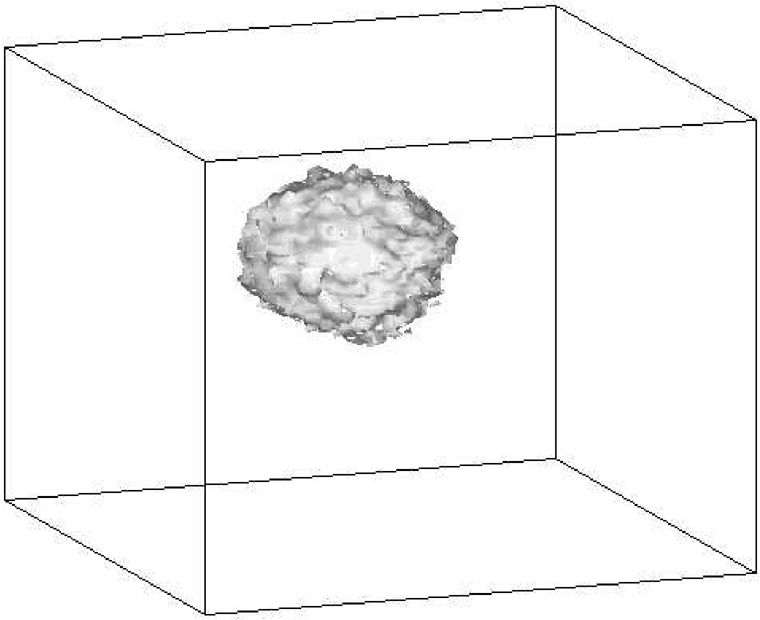


$$q \equiv \frac{g\phi_0^2}{4m^2}$$

Non-thermal Phase Transitions



String formation in the $\lambda(\phi_1^2 + \phi_2^2 - v^2)^2$ model,
$v \sim 10^{16}$ GeV.



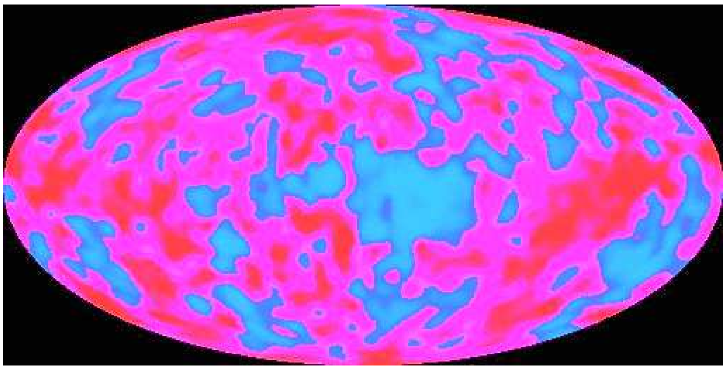
First order phase transition in the $\lambda(\phi^2 - v^2)^2 + g^2\phi^2 X^2$ model, $g^2/\lambda = 200$

Khlebnikov, Kofman, Linde & I.T. (98)

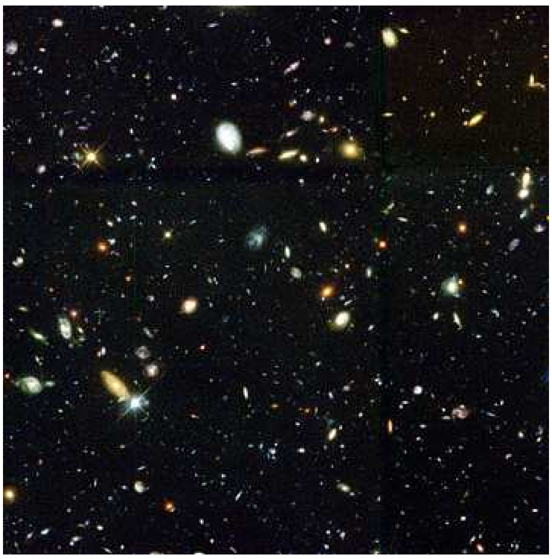
Gravitational creation of metric perturbations

Inflation

⇓



CMBR anisotropy
300,000 years after



LSS 15 billions years after

Inflationary perturbations

Assume Hubble parameter during inflation is constant,

$$a(\eta) = -\frac{1}{H\eta}$$

Mode functions of massles field $(\xi = 0)$ obey

$$\ddot{g}_k + k^2 g_k - \frac{2}{\eta^2} g_k = 0$$

Solutions with vacuum initial conditions

$$g_k = \frac{\mathrm{e}^{\pm ik\eta}}{\sqrt{2k}} \left(1 \pm \frac{i}{k\eta}\right)$$

After horizon crossing $k\eta \ll 1$

$$g_k = \pm \frac{i}{k\eta}, \quad \text{or} \quad \varphi = \mp \frac{iH}{k}$$

Field variance

$$\langle \varphi^2 \rangle = \frac{H^2}{(2\pi)^2} \int \frac{dk}{k}$$

Curvature perturbations

Spatial Curvature

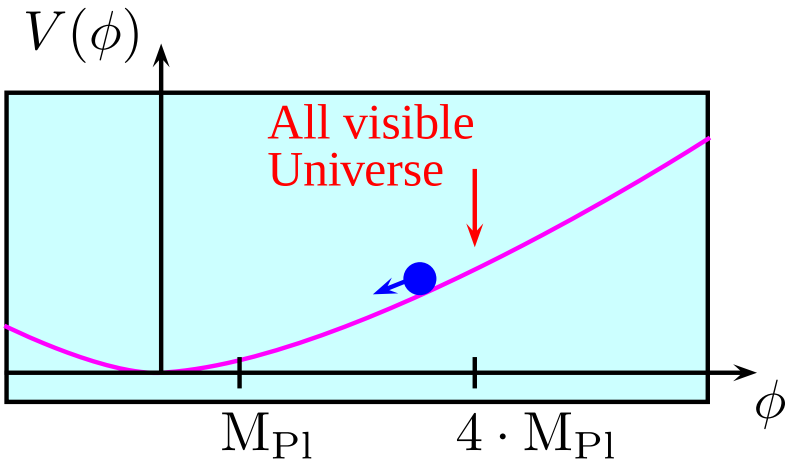
$$^{(3)}R \propto \frac{1}{a^2}$$

Its perturbation

$$\zeta \propto \frac{\delta a}{a} = H\delta t = H\frac{\delta\varphi}{\dot{\varphi}}$$

Since $\langle\varphi^2\rangle \sim H^2$ we have

$$\zeta_k \sim \frac{H^2}{\dot{\varphi}} \sim \frac{\delta\rho_k}{\rho} \sim P_k^{1/2}$$



- Cosmological scales encompass small $\Delta\phi$ interval
- Potentail should be flat over this range of $\Delta\phi$

$$\Downarrow$$

Observables essentially depend on a first few derivatives of V (slow roll parameters)

$$V(\phi_0)$$

$$\epsilon \equiv \frac{M_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2$$

$$\eta \equiv \frac{M_{\rm Pl}^2}{8\pi}\frac{V''}{V}$$

Power spectra of Scalar (curvature) and Tensor (gravity waves) perturbations

$$P(k)_S = \frac{1}{\pi\epsilon}\frac{H^2}{M_{\rm Pl}^2} \qquad \Rightarrow \qquad \frac{P(k)_T}{P(k)_S} = 16\epsilon$$
$$P(k)_T = \frac{16}{\pi}\frac{H^2}{M_{\rm Pl}^2}$$

Spectra can be approximated as power law functions

$$P(k)_S = P(k_0)_S \left(\frac{k}{k_0}\right)^{n-1}$$
$$P(k)_T = P(k_0)_T \left(\frac{k}{k_0}\right)^{n_T}$$

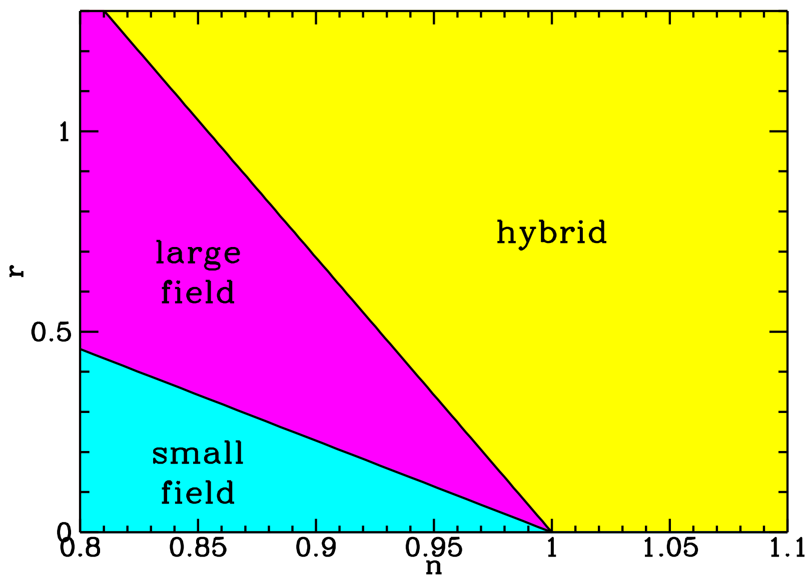
In slow roll parameters one finds

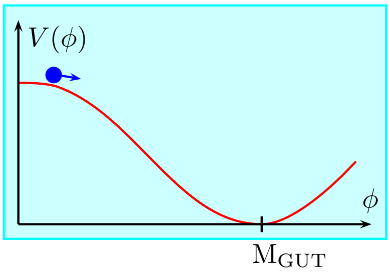
$$n - 1 = 2\eta - 6\epsilon$$
$$n_T = -2\epsilon$$

Consistency relation

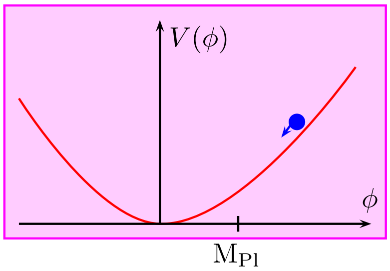
$$n_T \approx -\frac{1}{7}\, r \quad \text{where} \quad r \equiv \frac{C^T}{C^S}$$

Typical models of inflation occupy these regions of parameter space:

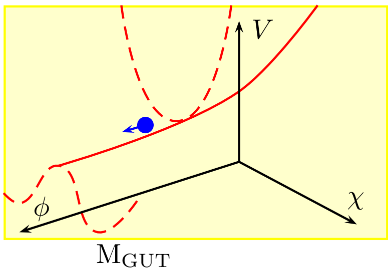




New inflation



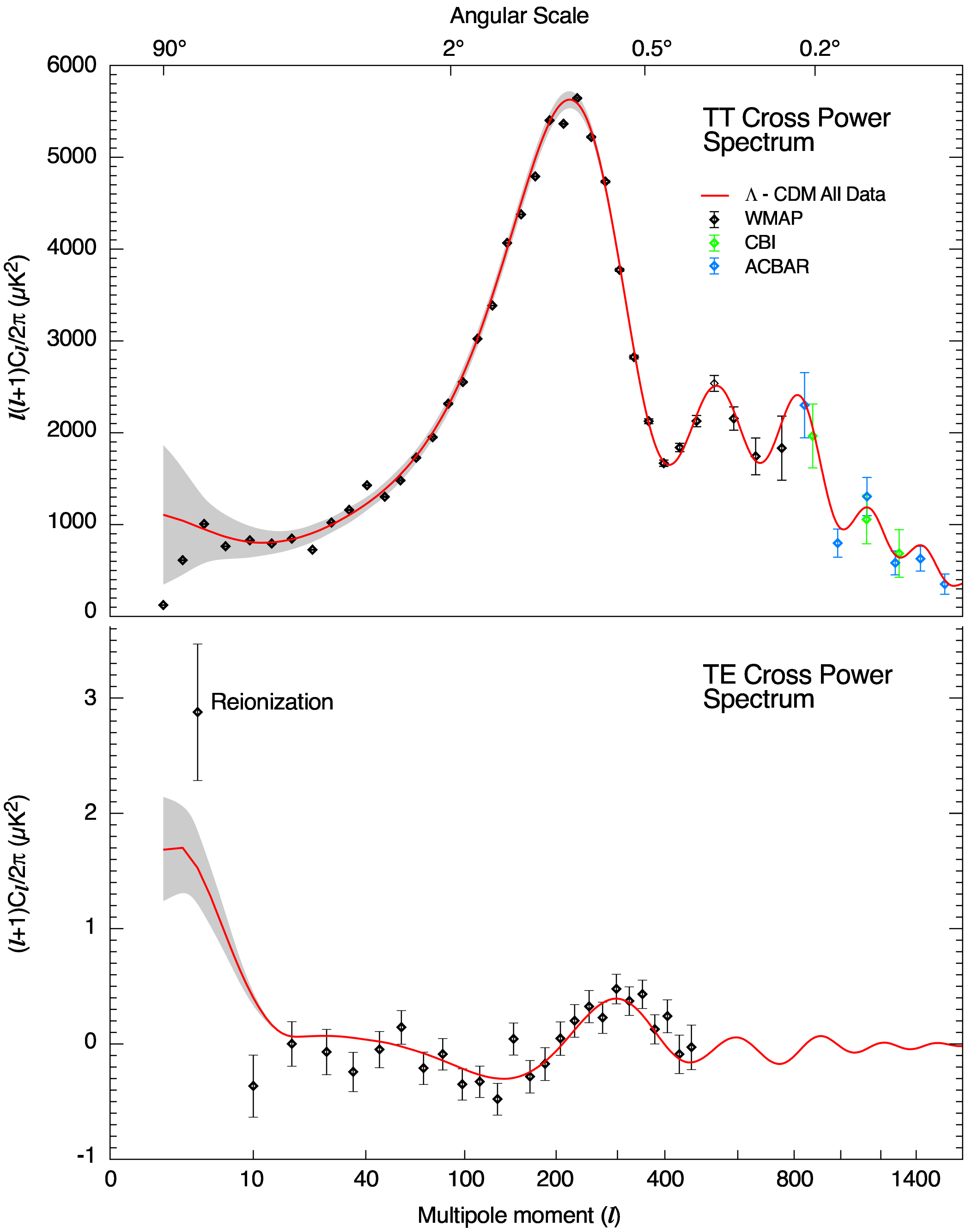
Chaotic inflation



Hybrid inflation

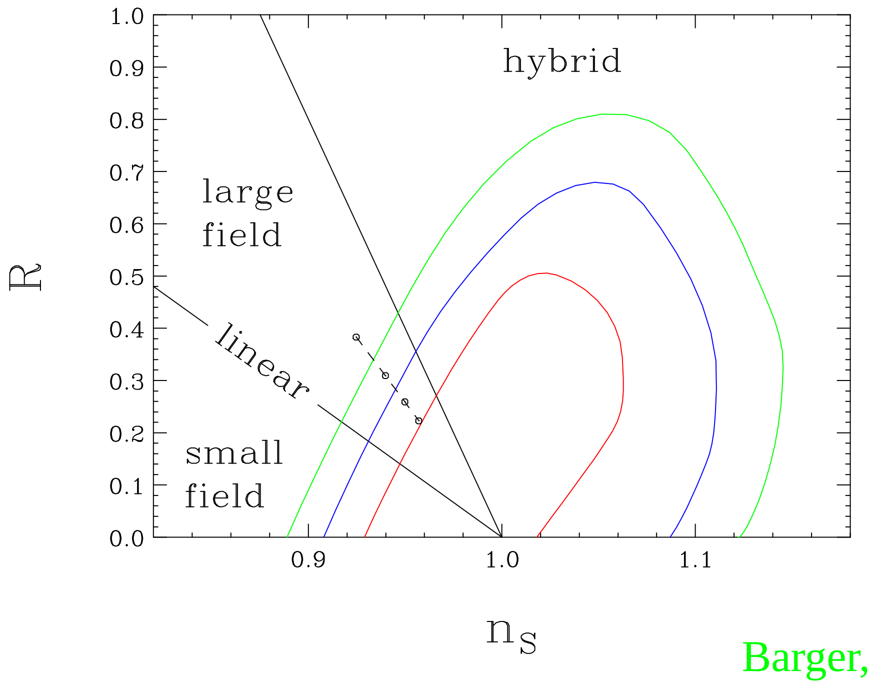
Dodelson, Kinney, Kolb (1997)

WMAP CMBR anisotropy spectrum



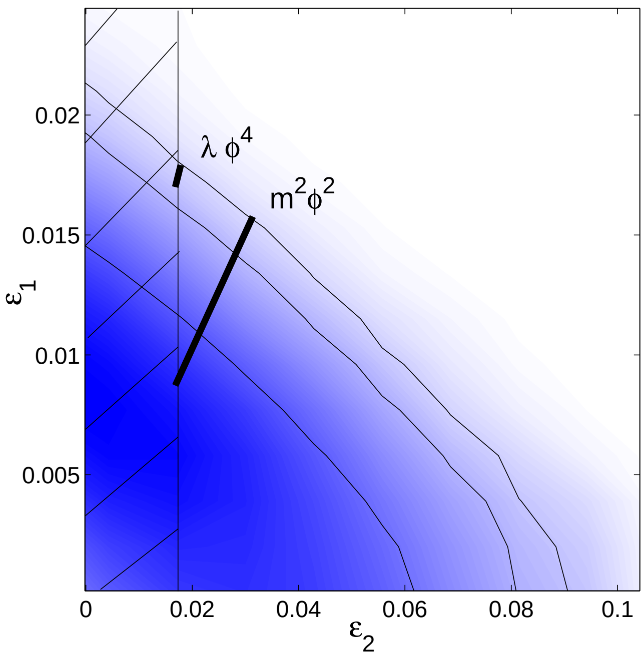
$$\Omega_0 = 1.0 \pm 0.03, \quad n_s = 0.99 \pm 0.04$$

Implications for Inflation



Barger, Lee and Marfatia (03)

For the $V \propto \phi^p$ chaotic inflation model this means

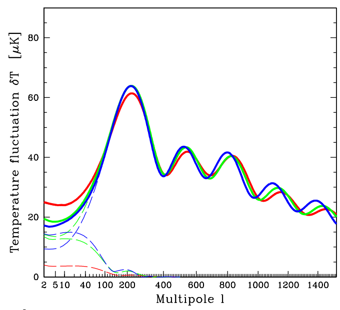


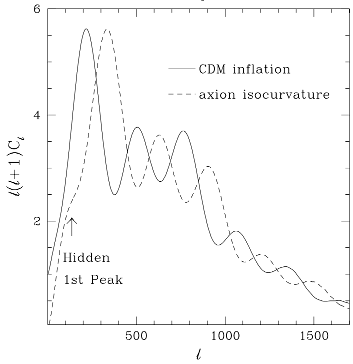
$$n_s - 1 = -2\epsilon_1 - \epsilon_2$$
$$R = 16\epsilon_1$$

Leach and Liddle (03)

TESTING INFLATION

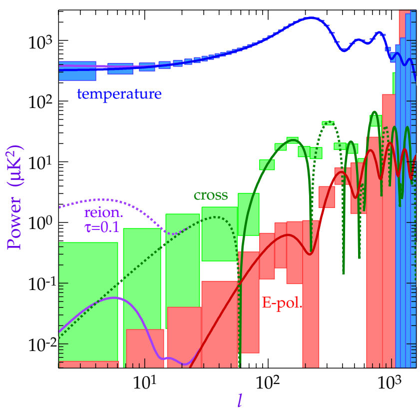
- $\Omega_0 = 1$ $(\star\star\star)$
- Nearly scale invariant $(\star\star)$ spectrum of
  - scalar $(\star\star)$ and
  - tensor $(?)$ perturbations
  - which are Gaussian $(\star)$
  - and of superhorizon scale $(\star)$
- Consistency relations $(?)$

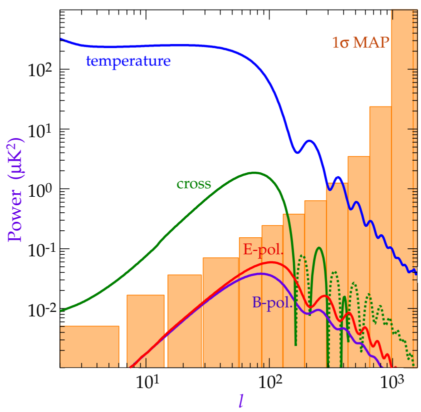




Future is bright

- High precision
- Polarization
- Tensor mode

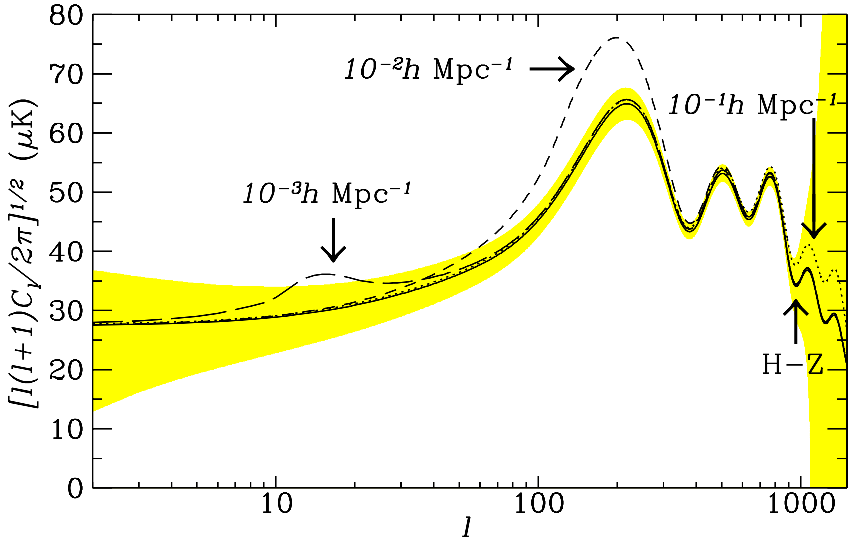


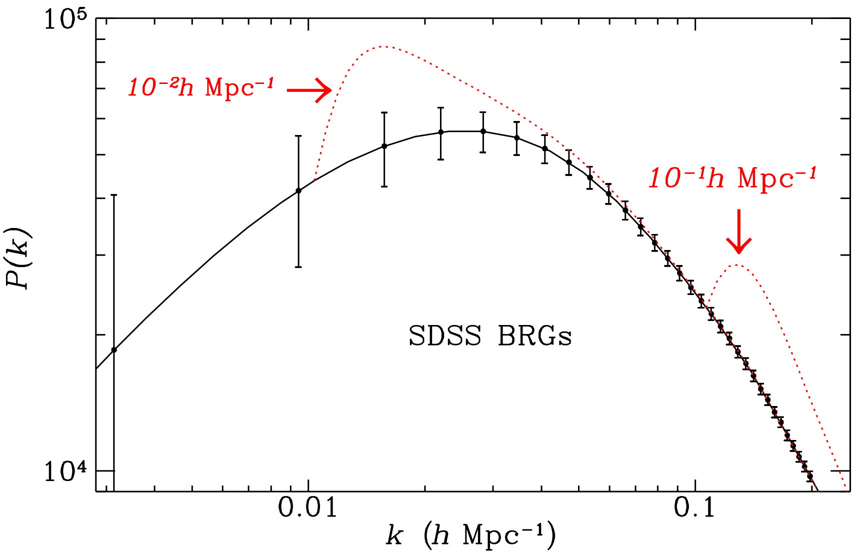


Hu & White (96)

Creation of matter during Inflation

Probe of Sub-Plankian particle content





Particles with $M_\psi \sim M_{\rm Pl}$ and coupling $g > 0.2/N^{2/5}$ are detectable.

Chung, Kolb, Riotto & I.T. (00)

Relativistic Turbulence

A Long Way from Preheating to Equilibrium

With R. Micha

Questions:

- How system approaches equilibrium ?
- When ? What is thermalization temperature ?

Important since it influences:

- Inflationary predictions
- Baryogenesis
- Abundance of gravitino and dark matter relics
- Is of general Statistical interest

Thermalization after Inflation

Outline:

- Lattice results
- Kinetic theory
  - Basics of turbulence
  - Driven turbulence
  - Decaying turbulence
  - Self-similar solutions
- Thermalization

Approach:

- Lattice simulations (as a guidance)
- Kinetic theory

Consider simplest $\lambda\varphi^4$ model

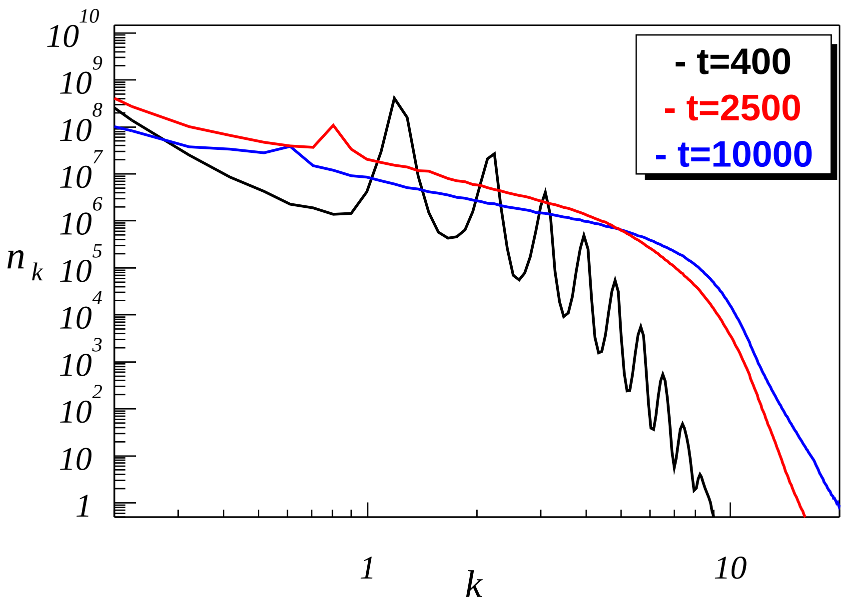
In conformal frame, $\phi = \varphi/a$, and rescaled coordinates, $x^\mu \to \sqrt{\lambda}\varphi(0)\, x^\mu$, the equation of motion

$$\Box\phi + \phi^3 = 0$$

can be solved on a lattice and various quantities be measured

- Zero mode, $\phi_0 = \langle\phi\rangle$
- Variance, $\langle\phi^2\rangle$ - ϕ_0^2
- Particle number, $n_k = \langle a^\dagger(k)a(k)\rangle$
- Correlators, $\langle aa\rangle$, $\langle a^\dagger a^\dagger aa\rangle$, $\langle\pi^2\rangle$, ...

Particle spectra on a lattice



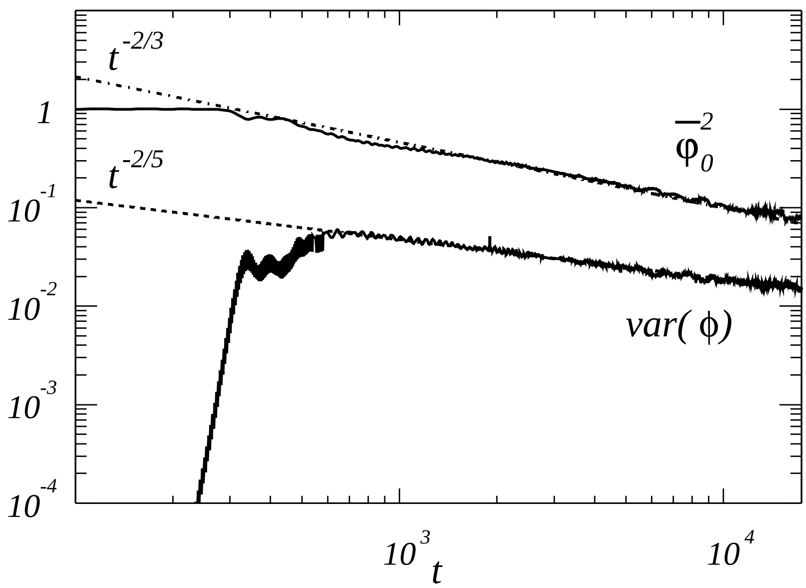
Complications:

- Insufficient dynamical range in k
- Hopelessly long integration time

Is it possible to use simple kinetic description ?

Complications:

- Zero mode never dies
- Occupation numbers too big
- Anomolous correlators are non-vanishing
- Not clear how to write collision integral

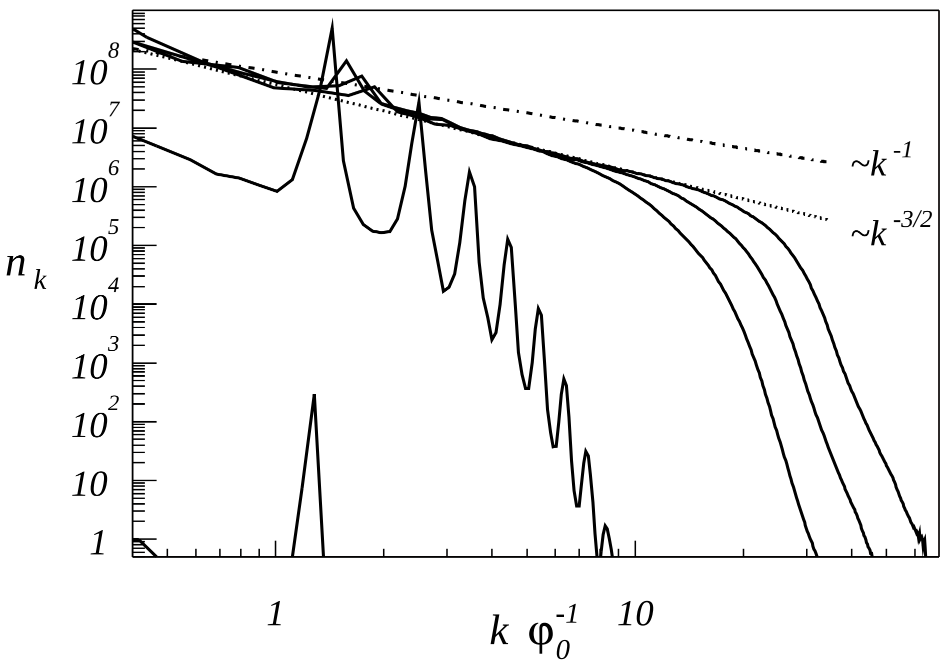


Hint

Re-scale the field and coordinates by the current amplitude of the zero mode

$$\Box\phi + \phi^3 = 0$$

Here $x^\mu \to x^\mu\,\phi_0$ and therefore $k \to k\,/\phi_0$



Let $n \sim k^{-\alpha}$.

Theory of stationary Kolmogorov turbulence predicts

- $\alpha = \frac{5}{3}$ for 4-particle interaction
- $\alpha = \frac{3}{2}$ for 3-particle interaction

Kinetic Theory

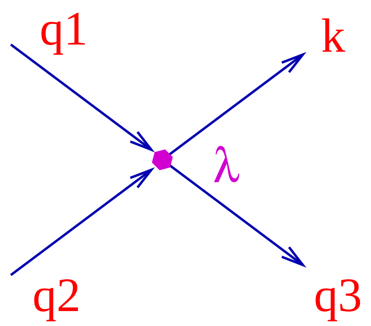
Kinetic equation

$$\dot{n}_k = I_k[n]$$

Collision integral

$$I_k[n] = \int d\Omega(k, q_i)\, F(k, q_i)$$

Example:



$$d\Omega(k, q_i) = \frac{(2\pi)^4 |M|^2}{2\omega_k} \delta^4(k_\mu, q_{i\mu}) \prod_{i=1}^{3} \frac{d^3 q_i}{2\omega_i (2\pi)^3}$$

In full quantum problem

$$F(k, q_i) = (1+n_k)\,(1+n_{q_1})n_{q_2}n_{q_3} - n_k n_{q_1}(1+n_{q_2})\,(1+n_{q_3})$$

For classical waves ($n \gg 1$)

$$F(k, q_i) = (n_k + n_{q_1})n_{q_2}n_{q_3} - n_k n_{q_1}(n_{q_2} + n_{q_3})$$

Scaling

Rescaling of n

$$F(\zeta n) = \zeta^{m-1} F(\zeta n) \ ,$$

where m is the number of particles which participate in the process.

Rescaling of momenta

$$d\Omega(\xi k, \xi q_i) = \xi^{\mu} \, d\Omega(k, q_i) \ ,$$

where μ depends upon theory and number of dimensions. E.g. $\mu = 1$ for a relativistic theory with dimensionless couplings in $d = 3$.

If $n(q) \propto q^{-s}$ we also have

$$F(\xi k, \xi q_i) = \xi^{\,-s\,(m-1)} \, F(k, q_i) \ .$$

This gives e.g.

$$I_{\xi k}[n] = \xi^{-\nu} I_k[n] \ ,$$

where $\nu = \mu - s\,(m-1)$

Kolmogorov Turbulence

We have a source of energy (or particles) located at $k = k_i$ and a sink located at $k = k_f$. Energy conserves

$$\partial_t(\omega_k \, n_k) + \nabla_k \cdot j_k = 0.$$

In statiory situation energy flux is constant through any surface

$$\begin{aligned} \text{Flux} &= -\int^p d^dk \, \omega_k \, \dot{n}_k = -\int^p dk \, k^{d-1} \omega_k \, I_k[n] \\ &\propto -\, p^{d+\alpha-\nu} \frac{I_1(\nu)}{d+\alpha-\nu} \, , \end{aligned}$$

where $\omega(\xi k) = \xi^\alpha \omega(k)$.

We find $\nu = d + \alpha$ or

$$s = \frac{d+\alpha+\mu}{m-1}$$

Self-similar evolution

Assume $n(k, \tau) = A^{\gamma}\, n_0(kA) \equiv A^{\gamma}\, n_0(\zeta)$
where $\tau \equiv t/t_0$ and $A = A(\tau)$.

With this anzats kinetic equation separates into two equations. The first one determines the shape of a distribution function:

$$\gamma n_0 + \zeta \frac{dn_0}{d\zeta} = -I(\zeta)$$

The second one fixes its evolution, and has a solution

$$A(\tau) = \tau^{-p} \quad \text{where} \quad p = \frac{1}{\gamma(m-2) - \mu}$$

Two important cases

- Isolated system (energy conserves), $\gamma = (d + \alpha)$ and

$$p_i = \frac{1}{(d+\alpha)(m-2) - \mu}$$

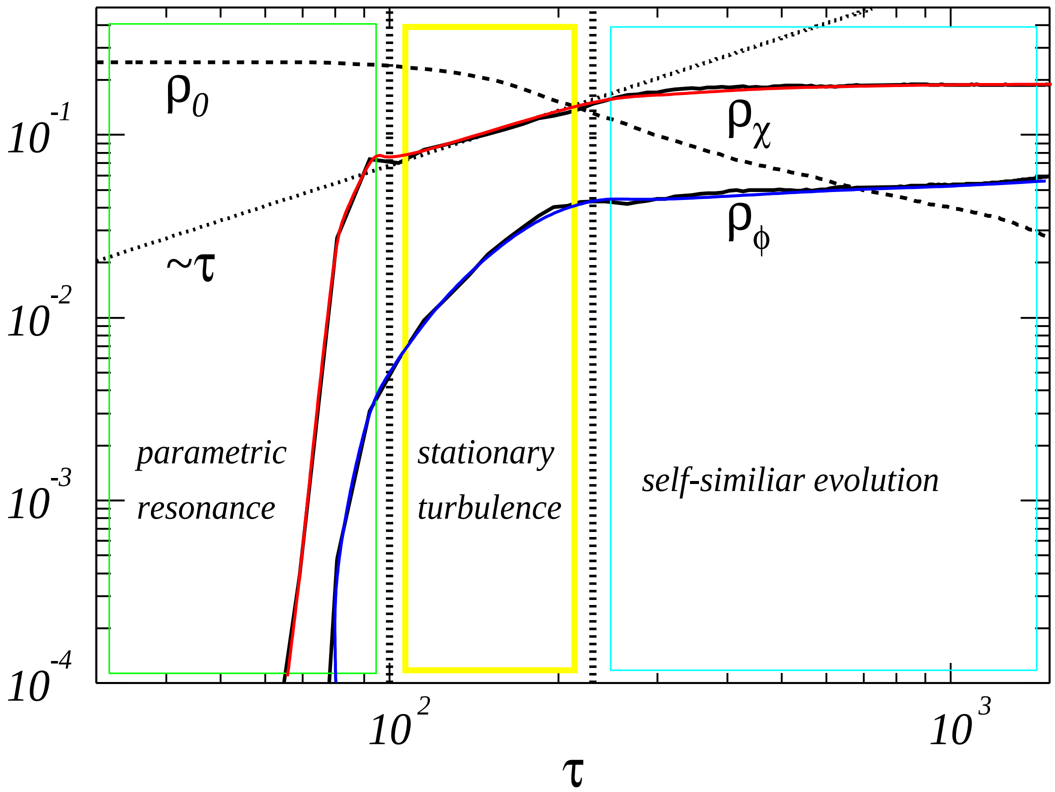
- Stationary source, $\gamma = s$ and

$$p_t = 3\, p_i$$

Three major epochs of reheating

$$V(\chi, X) = \frac{\lambda_\phi}{4}\phi^4 + \frac{g}{2}\phi^2\chi^2 + \frac{\lambda_\chi}{4}\chi^4$$

At large g/λ_ϕ and/or large $\lambda_\chi/\lambda_\phi$ parametric resonance stops when n_χ are low



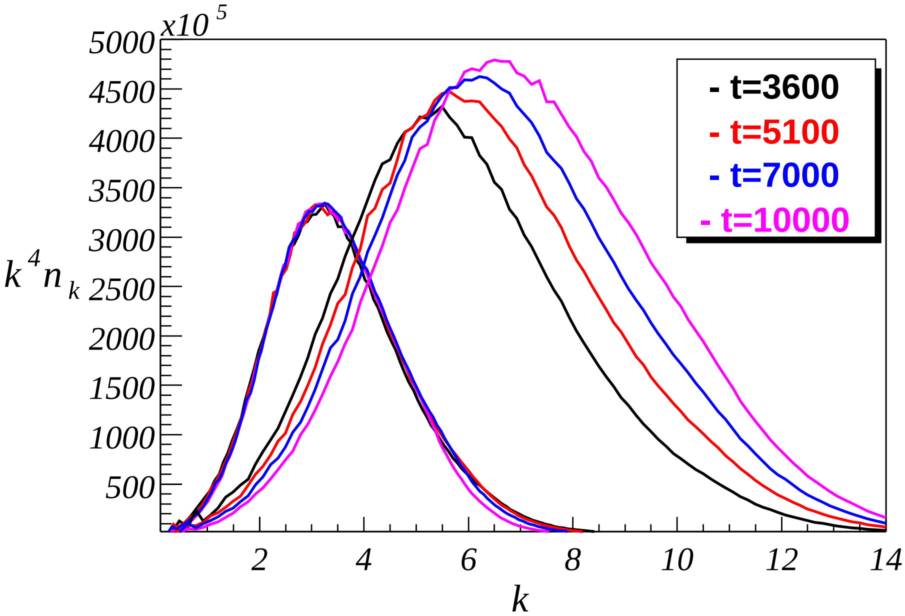
$$q = 30\lambda_\phi, \qquad \lambda_\chi = 300\lambda_\phi$$

Decaying turbulence

At late times we expect self-similarity with conserved energy

$$n(k,t) = t^{-q}\, n_0(kt^{-p})$$

Excellent fit to numerical data: $q = 3.5p$ and $p = \frac{1}{5}$



For $\lambda\phi^4$ model and 4-particle interaction in $d = 3$ we have

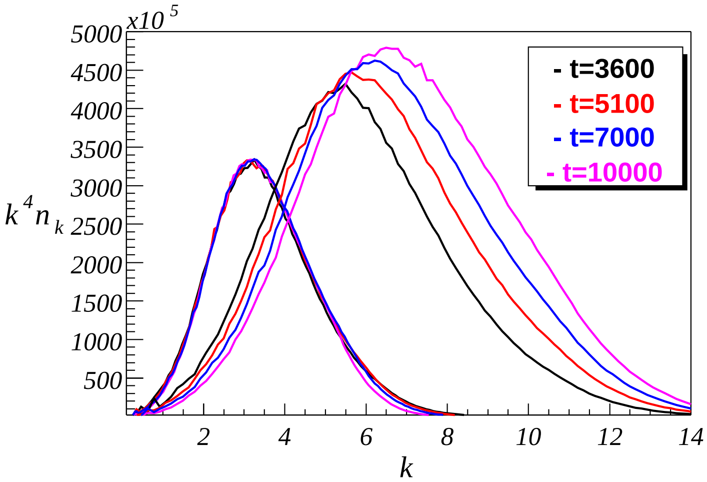
(1) Assuming energy conservation: $q = 4p$ and $p = \frac{1}{7}$

(2) Assuming stationary turbulence: $q = \frac{5}{3}\,p$ and $p = \frac{3}{7}$

Truth is in between. Correcting for the energy influx from the zero mode we get $p = \frac{1}{6}$.

Thermalization

At late times influence of the zero mode should become negligible and $p = \frac{1}{7}$. This exponent determines the rate with which a system approaches equilibrium



$k_{\max}(\tau) = k_0\,\tau^p$, where $k_0 = \lambda^{1/2}\varphi_0$.

Thermalization will occur when $k_{\max}^4 \sim T^4 \sim \lambda\varphi_0^4$.

Time to thermalization $\tau \sim \lambda^{-7/4} \sim 10^{21}$.

Scale factor in comoving coordinates $a(\tau) = \tau$ and we find for thermalization temperature

$$T \sim \frac{k_{\max}}{a(\tau)} = \lambda^2\varphi_0 = 10^{-26} M_{\rm Pl} = 100\ \text{eV}\,.$$

One can use "naive" perturbation theory to estimate thermalization tempreature