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Recursive numerical calculus of one-loop tensor integrals

The motivation

• Computing one-loop corrections gets complicated when the number of the particles undergoing the scattering process grows

$\Rightarrow$ List of problems

a) Large number of Feynman Diagrams (FD)

b) Reduction of tensor integrals to scalar ones

c) Control over the numerical inaccuracies.

• More than the large number of FD what really renders the calculation difficult is the fact that *each diagram requires a lot of work to be computed*, that is related to the second issue of the list.

• Ideally, the perfect situation would be such that one simply writes down the amplitude

$$A^{(n)} = \sum_{m,i,j} I^{(n)}_{m;\,\mu_1\cdots\mu_i}(\{p_j\},\{m_j\})\; S^{\mu_1\cdots\mu_i}_{m,j}\,, \tag{1}$$

and all the rest is handled by a numerical program.

• We present a way to reduce, *numerically and recursively*, the tensor integrals

$$I^{(n)}_{m;\,\mu\cdots\nu}(\{p_j\},\{m_j\}) \equiv \int d^n q \frac{q_\mu \cdots q_\nu}{D_0 D_1 \cdots D_m}\,,$$
$$D_j = (q+p_j)^2 - m_i^2\,, \quad p_0 = 0 \tag{2}$$

to the *standard set* of scalar one-loop integrals.

• Our method is applicable, irrespective of the number of external legs, to any configuration of internal and/or external scales.

• We can control ultraviolet and infrared/collinear divergences *without explicitly subtracting them.*

• Part of the Gram determinant fake singularities compensate in such a way that, at each step of the iteration, *only inverse square roots of Gram determinants appear*, improving the numerical stability of the calculation.

• In addition, we have studied all possible sources of numerical instabilities, *giving general prescriptions on how to cure them.*

• See `hep-ph/0404120` for more details.

The method in 4 dimensions

• We express two among the vectors $q_\mu \cdots q_\nu$ appearing in the numerator of Eq. (2) directly in terms of denominators, *before* performing the actual integration

$$q_\mu q_\nu \quad = \quad D^\lambda_{(a)}\, q^\sigma T_{\mu\nu\lambda\sigma} + D_{(b)}\, T_{\mu\nu} + D_{(c)}\, q^\lambda T_{\mu\nu\lambda}\,, \qquad (3)$$

where $D^\lambda_{(a)}$, $D_{(b)}$ and $D_{(c)}$ are linear combinations of the denominators D_0, D_1, D_2, D_3 and where $T_{\mu\nu\lambda\sigma}$, $T_{\mu\nu\lambda}$, $T_{\mu\nu}$ only depend on three linearly independent momenta in the set $\{p_i\}$ $(i = 1 : m)$.

• A recursion relation simply follows by dividing Eq. (3) by the $(m+1)$ denominators of Eq. (2), multiplying by an arbitrary number of 4-vectors $q_\rho \cdots q_\tau$ and integrating over d^4q.

• That can be achieved by decomposing each q in the product $q^\mu q^\nu$ in terms of four judiciously chosen massless vectors

$$q^\mu = \sum_{i=1}^{4} c_i\, {\ell_i}^\mu\,, \qquad {\ell_i}^2 = 0\,. \tag{4}$$

where $(\omega^- = (1-\gamma_5)/2)$

$$\begin{aligned} p_1^\mu &= {\ell_1}^\mu + \alpha_1 {\ell_2}^\mu, & p_2^\mu &= {\ell_2}^\mu + \alpha_2 {\ell_1}^\mu \\ {\ell_3}^\mu &= \bar{v}(\ell_1)\,\gamma^\mu\,\omega^-\,u(\ell_2), & {\ell_4}^\mu &= \bar{v}(\ell_2)\,\gamma^\mu\,\omega^-\,u(\ell_1). \end{aligned} \tag{5}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} q^\mu \quad &= \quad \frac{1}{2\,(\ell_1\cdot\ell_2)}\left[D^\mu - (q\cdot\ell_4)\,\frac{{\ell_3}^\mu}{2} - (q\cdot\ell_3)\,\frac{{\ell_4}^\mu}{2}\right] \\ D^\mu \quad &\equiv \quad 2\,(q\cdot\ell_2)\,{\ell_1}^\mu + 2\,(q\cdot\ell_1)\,{\ell_2}^\mu \end{aligned}}$$

• In the term D^μ, denominators can be immediately reconstructed by using

$$2\,(q \cdot p_i) \quad = \quad \left[D_i - D_0 + m_i^2 - m_0^2 - p_i^2\right] \, . \tag{6}$$

• By applying this decomposition also to q^ν, terms such $(q \cdot \ell_3)^2$, $(q \cdot \ell_4)^2$ and $(q \cdot \ell_3)(q \cdot \ell_4)$ appear. With the help of the Dirac equation and the completeness relations for massless spinors $\bar{v}(\ell_i)$ and $u(\ell_i)$, they can also be expressed in terms of denominators

$$\begin{aligned}
(q \cdot \ell_3)(q \cdot \ell_4) \quad &= \quad q^\alpha D_\alpha - \gamma q^2 \\
(q \cdot \ell_{3(4)})(q \cdot \ell_{3(4)}) \quad &= \quad \frac{1}{(p_3 \cdot \ell_{4(3)})} \left\{ \left[\gamma q^2 - q^\alpha D_\alpha\right] (p_3 \cdot \ell_{3(4)}) \right. \\
&- \quad \left. \left[\gamma\,(q \cdot p_3) - p_3^\alpha D_\alpha\right] (q \cdot \ell_{3(4)}) \right\} \, .
\end{aligned} \tag{7}$$

• Collecting all terms one gets

$$
\begin{aligned}
q_\mu q_\nu &= \frac{1}{2\gamma}\left[D^\lambda q^\sigma\right] T_{\mu\nu\lambda\sigma} - \frac{1}{4\,\gamma}\left[D_0 + m_0^2\right] T_{\mu\nu} \\
&- \frac{1}{4\,\gamma}\left[\left((D_3 - D_0 + f_{30}) - \frac{2}{\gamma} p_{3\alpha}\, D^\alpha\right) q^\lambda\right] T_{\mu\nu\lambda}\,, \qquad (8)
\end{aligned}
$$

where

$$
\begin{aligned}
T_{\mu\nu\lambda} &= \frac{\ell_{3\mu}\,\ell_{3\nu}\,\ell_{4\lambda}}{(\ell_3 \cdot p_3)} + \frac{\ell_{4\mu}\,\ell_{4\nu}\,\ell_{3\lambda}}{(\ell_4 \cdot p_3)} \\
T_{\mu\nu} &= t_{\mu\nu} - p_3^\sigma T_{\mu\nu\sigma} \\
t_{\mu\nu} &= \ell_{3\mu}\ell_{4\nu} + \ell_{4\mu}\ell_{3\nu} \\
T_{\mu\nu\lambda\sigma} &= g_{\mu\lambda}\left(g_{\nu\sigma} - \frac{t_{\nu\sigma}}{2\gamma}\right) + g_{\nu\lambda}\left(g_{\mu\sigma} - \frac{t_{\mu\sigma}}{2\gamma}\right) + \frac{g_{\lambda\sigma}}{2\gamma}\, T_{\mu\nu} \\
\gamma &= 2\,(\ell_1 \cdot \ell_2)\,. \qquad (9)
\end{aligned}
$$

- The corresponding recursion relation then reads

$$
\begin{aligned}
I_{m;\,\mu\nu\rho\cdots\tau} &= \frac{\beta}{2\gamma} T_{\mu\nu\lambda\sigma} \left\{ J^{\lambda\sigma}_{m;\,\rho\cdots\tau} \right\} \\
&- \frac{1}{4\,\gamma} T_{\mu\nu} \left\{ m_0^2\, I_{m;\,\rho\cdots\tau} + I_{m-1;\,\rho\cdots\tau}(0) \right\} \\
&- \frac{1}{4\,\gamma} T_{\mu\nu\lambda} \left\{ f_{30}\, I^{\lambda}_{m;\,\rho\cdots\tau} + I^{\lambda}_{m-1;\,\rho\cdots\tau}(3) \right. \\
&- \left. I^{\lambda}_{m-1;\,\rho\cdots\tau}(0) - \frac{2\,\beta}{\gamma} p_{3\alpha}\, J^{\alpha\lambda}_{m;\,\rho\cdots\tau} \right\}, \qquad (10)
\end{aligned}
$$

where $\beta = 1/(1-\alpha_1\alpha_2)$ and

$$
\begin{aligned}
J^{\lambda\sigma}_{m;\,\rho\cdots\tau} &= \left(f_{10} r_2^{\lambda} + f_{20} r_1^{\lambda}\right) I^{\sigma}_{m;\,\rho\cdots\tau} + r_2^{\lambda}\, I^{\sigma}_{m-1;\,\rho\cdots\tau}(1) \\
&+ r_1^{\lambda}\, I^{\sigma}_{m-1;\,\rho\cdots\tau}(2) - \left(r_1^{\lambda} + r_2^{\lambda}\right) I^{\sigma}_{m-1;\,\rho\cdots\tau}(0) \\
I_{m-1;\,\mu\cdots\nu}(i) &= \int d^4q \frac{q_\mu \cdots q_\nu}{D_0 D_1 \cdots D_{i-1} D_{i+1} \cdots D_m}\,. \qquad (11)
\end{aligned}
$$

The method in $n = 4 + \epsilon$ dimensions

- We reduce the problem to 4-dimensional tensors, *that we know how to handle*, by splitting n-dimensional objects as follows:

$$\bar{q} = q + \tilde{q}\,, \quad \bar{q}^2 = q^2 + \tilde{q}^2\,.$$

- The rank k n-dimensional tensors we want to evaluate are

$$I^{(n)}_{m;\,\bar{\mu}_1\cdots\bar{\mu}_k} = \int d^n\bar{q}\,\frac{1}{\bar{D}_0\cdots\bar{D}_m}\prod_{a=1}^{k}\bar{q}_{\mu_a}\,, \qquad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2\,.$$

- Then

$$\begin{aligned}\prod_{a=1}^{k}\bar{q}_{\mu_a} &= \prod_{a=1}^{k} q_{\mu_a} + \sum_{b=1}^{k}\tilde{q}_{\mu_b}\prod_{a\neq b}^{k} q_{\mu_a} \\ &+ \sum_{b>c}^{k}\tilde{q}_{\mu_b}\tilde{q}_{\mu_c}\prod_{a\neq b,c}^{k} q_{\mu_a} + \cdots + \prod_{a=1}^{k}\tilde{q}_{\mu_a}\,. \qquad (12)\end{aligned}$$

• The 4-dimensional term can be worked out as described before while the remaining terms give rise to extra integrals involving $\tilde{q}^{2\ell}$:

$$I^{(n;\,2\ell)}_{m;\,\mu_1\cdots\mu_h} = \int d^n\bar{q}\,\frac{\tilde{q}^{2\ell}}{\bar{D}_0\cdots\bar{D}_m}\prod_{a=1}^{h} q_{\mu_a}\,, \tag{13}$$

that are very easy to evaluate $\mathcal{O}(1)$.

• For example:

$$I^{(n;\,2(m-1+d-s))}_{m;\,\mu_1\cdots\mu_{2s}} = \begin{cases} -i\pi^2\,\frac{g_{\mu_1\cdots\mu_{2s}}}{2^s(s+1)!}\,\frac{\Gamma(m-1)}{\Gamma(m+1)} + \mathcal{O}(\epsilon) & \text{if } d=0 \\ 0 & \text{if } d<0 \end{cases}$$

where d is the superficial degree of divergence of the extra integral.

• To summarize, any n-dimensional one-loop amplitude

$$A^{(n)} = \sum_{m,i,j} I^{(n)}_{m;\,\bar{\mu}_1\cdots\bar{\mu}_i}(\{p_j\},\{m_j\})\; S^{\bar{\mu}_1\cdots\bar{\mu}_i}_{m,j} \tag{14}$$

can be computed following 4 steps:

1- Split n-dim. tensors into 4-dim. part plus extra integrals;

2- Compute the 4-dimensional part numerically and iteratively starting from the *standard* set of one loop scalar functions;

3- Calculate $\mathcal{O}(1)$ the needed extra integrals;

4- Contract numerically the computed loop-tensors with the external tensors $S_{m,j}$.

• *All divergences are, at the end, contained in the pole parts of the scalar integrals, parts to which one can give any value to numerically check all relevant cancellations.*

Particular cases

• The rank of the tensor integral in the l.h.s. of Eq. (10) should be at least 2 and $m \geq 3$, otherwise there are not enough momenta to perform the decomposition neither denominators to cancel the reconstructed ones.

• Those two particular cases can also be handled by using the fact that some term do not contribute *after* integration. For example:

$$\int d^n\bar{q}\,\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}(q\cdot\ell_3)^k = 0$$

$$\int d^n\bar{q}\,\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}(q\cdot\ell_4)^k = 0\,, \qquad \forall\, k = 1,2,3\cdots$$

Collinear and coplanar configurations

• Singular kinematical configurations give a smoother behaviour than in other approaches. For example, in Eq. (10)

$$\beta = \frac{p_1^2}{2\alpha_1\sqrt{\Delta_{12}}} \qquad \text{where} \quad \Delta_{12} = (p_1 \cdot p_2)^2 - p_1^2 p_2^2 \,. \tag{15}$$

• The same pattern is found by analyzing all other "dangerous" denominators of Eq. (10) in the limit of exceptional momenta.

• When Δ_{12} is *exactly* zero one simply starts the reduction by using $p_2 = \lambda p_1$ from the beginning.

• This same technique allows one to build up reliable approximations in the regions where $\Delta_{12} \sim 0$ and it is completely equivalent to extract the constant terms of a Taylor expansion in Δ_{12}, but *without the need to explicitly develop one loop tensor functions*.

Summary

• We have presented a way to compute, numerically and recursively, tensor integrals appearing in one-loop calculations relevant for the next generation of e^+e^- and pp Colliders.

• Our method is applicable, irrespective of the number of external legs, to any configuration of internal and/or external scales, and only requires the knowledge of the standard set of scalar one-loop integrals.

• Kinematical singularities give a smoother behaviour than in other approaches, and building up approximations around the regions of exactly vanishing Gram determinants do not requires explicit Taylor expansions.

• A code implementing the proposed method will be made available in the near future.