Comparison of Exact Results for the Virtual Correction to Bremsstrahlung in e^+e^- Annihilation at High Energies

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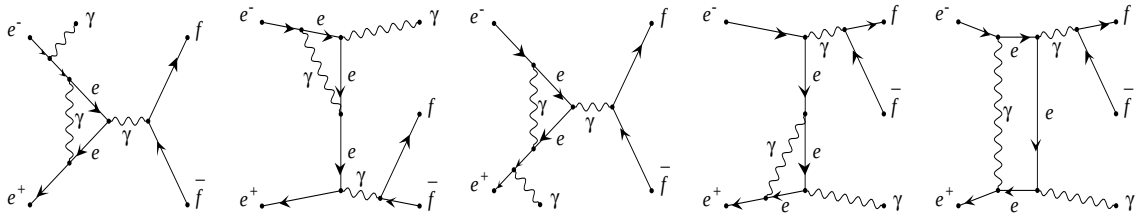
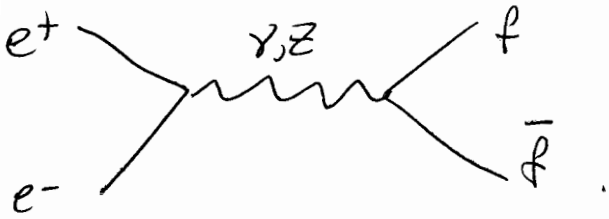
We have compared the virtual corrections to $e^+e^- \to f\overline{f} + \gamma$ as calculated by S. Jadach, M. Melles, B.F.L. Ward and S.A. Yost to several other expressions. The most recent of these comparisons is to the Leptonic tensor calculated by J.H. Kühn and G. Rodrigo for radiative return. Agreement is found to within 10^{-5} or better, as a fraction of the Born cross section. The massless limits of the results are all found to be analytically identical at NLL order.



In the 1990's, S. Jadach, M. Melles, B.F.L. Ward & S.A. Yost calculated all 2 photon real and virtual corrections to the $e^+e^- \to e^+e^-$ small angle Bhabha scattering process used in the Luminosity Monitor at LEP (and SLC). These calculations were incorporated in a Monte Carlo, and used to achieve a 0.06% precision tag for the program BHLUMI which calculated the luminosity process.

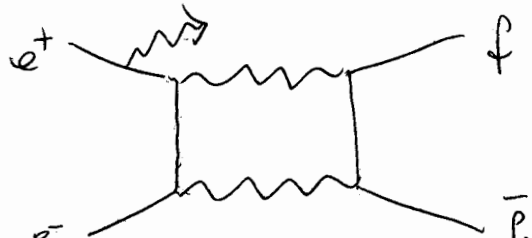
These processes were calculated in a small-mass approximation, which treats the external $e^{\pm}$ as massless, and then adds the most important mass corrections.

Once the Bhabha processes were obtained, the same functions could be used for another important LEP process, $e^+e^- \to f\bar{f}$ $(f \neq e)$, which is related to Bhabha scattering by "crossing":



Thus we have all 2-γ radiative corrections for this process as well, at small angles.

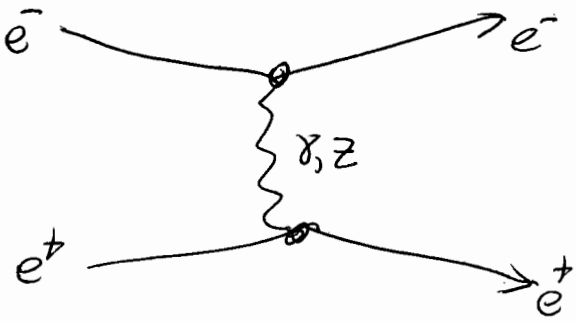
Only the "box diagrams" are omitted. They become important at larger angles, only.

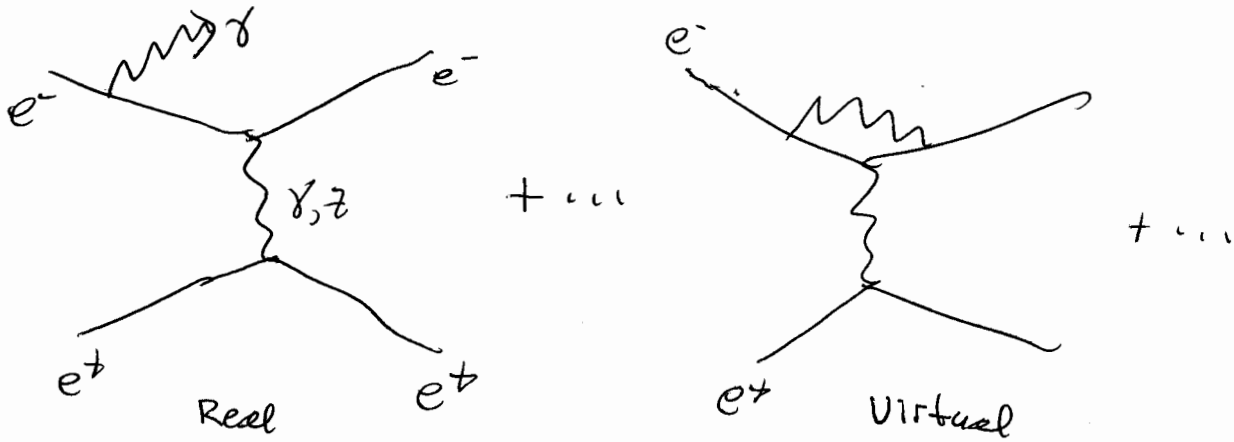


$+ \ldots$ (A calculation is in progress)

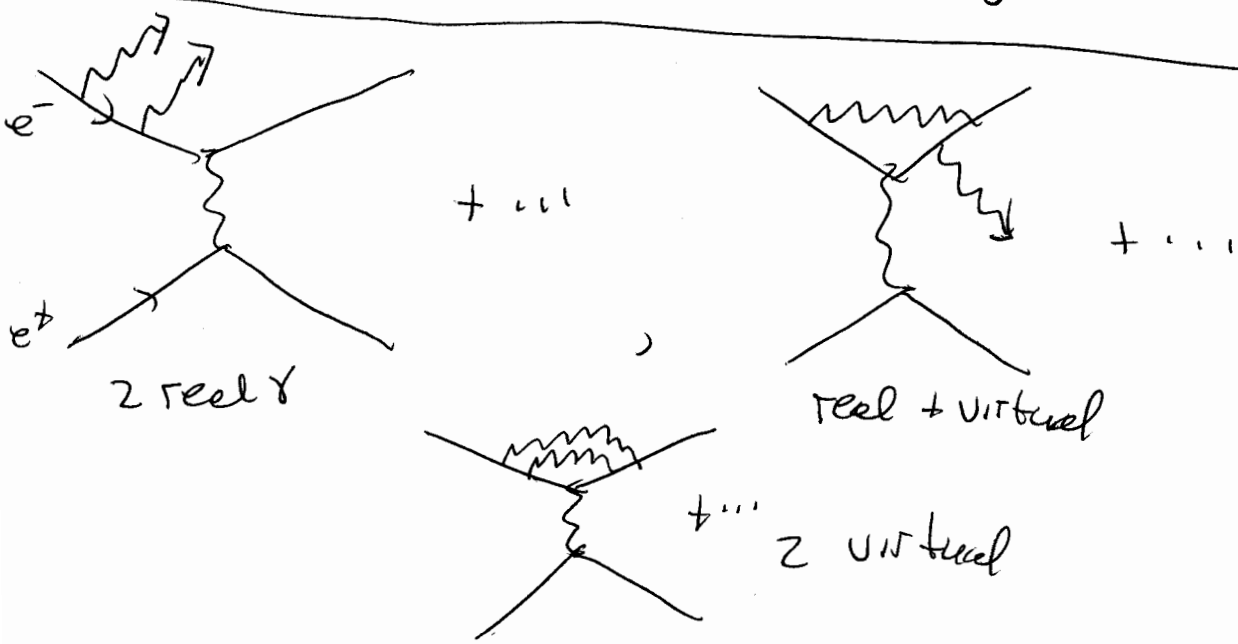
For the luminosity process, the graphs of interest were:

Basic Bhabha Scattering



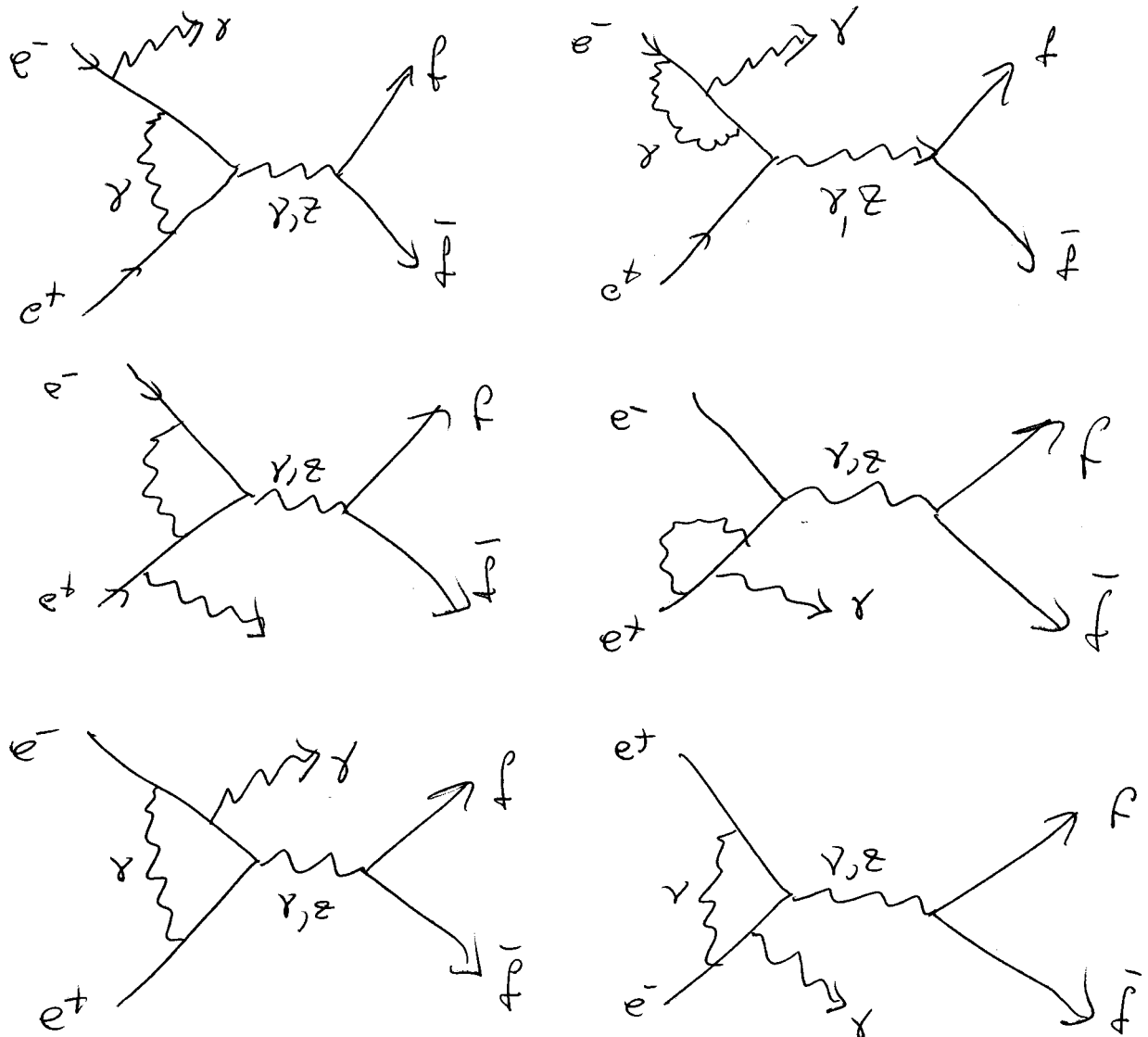


Order α corrections (Single-photon)



Order α^2 corrections (2-photon)

It is important to cross-check results with other relevant calculations. In this presentation, we will concentrate on the 1 real + 1 virtual photon radiative corrections to e^+e^- annihilation: specifically, the initial state radiation graphs



These graphs were calculated by Jadach, Melles, Ward & Yost in Phys Rev D 65, 073030 (2002) based on earlier results for the corresponding Bhabha graphs, in Jadach, Melles, Ward & Yost, Phys. Lett. B377, 168 (1996). The results were obtained using

- Helicity Spinor methods
- The algebraic manipulation program FORM
- The FF package of Scalar 1-loop Feynman integrals.

The sum of these amplitudes could be expressed as

$$\mathcal{M}_1^{ISR(1)} = \frac{e^2}{16\pi^2} \mathcal{M}_1^{ISR(0)} \left(f_0 + f_1 I_1 + f_2 I_2\right)$$

where $\mathcal{M}_1^{ISR(0)}$ is the 1-real-photon amplitude, I_1, I_2 are combinations of helicity spinors, and $f_{0,1,2}$ are scalar form factors.

In detail,

$$\mathcal{M}_1^{ISR(0)} = i\sigma e^3 G_{\lambda_1\lambda_3}(s') I_0 \frac{2\langle p_3,\sigma|p_4,-\sigma\rangle}{\langle p_1,-\sigma|k,\sigma\rangle\langle p_2,-\sigma|k,\sigma\rangle}$$

with kinematic variables

	Momentum	Helicity	
incoming e^-	p_1	λ_1	
incoming e^+	p_2	λ_2	$(=-\lambda_1)$
outgoing f	p_3	λ_3	
outgoing $\bar{f}$	p_4	λ_4	$(=-\lambda_3)$
radiated γ	k	σ	

$|p,\lambda\rangle$ = massless helicity spinor,

$$\not{p}|p,\lambda\rangle = 0, \quad \gamma_5|p,\lambda\rangle = \lambda|p,\lambda\rangle.$$

$$G_{\lambda,\lambda'}(z) = \frac{1}{z}\left\{1 + \frac{z(V_e+A_e\lambda)(V_f+A_f\lambda')}{z-M_Z^2+iz\Gamma_Z/M_Z}\right\}$$

is the propagator for γ, Z exchange,

$$I_0 = -\sqrt{2}\lambda_1\lambda_3\left(\langle h,-\sigma|h',\sigma\rangle\right)^2$$

where $h = \begin{cases} p_2 & \text{if } \sigma=\pm\lambda_1 \\ p_1 & \end{cases}$, $h' = \begin{cases} p_3 & \text{if } \sigma=\pm\lambda_3. \\ p_4 & \end{cases}$

The dominant term in $\mathcal{M}_1^{\Sigma SR(1)}$ is proportional to $\mathcal{M}_1^{\Sigma SR(0)}$: the f_0 term,

$$f_0 = 4\pi B_{YFS}(s, m_e) + 2(L-1-i\pi) + \frac{r_2}{1-r_2}$$

$$+ \frac{x}{r_1(1-r_2)} R(r_1, r_2) + R(r_2, r_1) + \left[3 + \frac{2r_2}{x(1-r_2)}\right] \ln(1-x)$$

$$- \frac{r_2(2+r_1)}{(1-r_1)(1-r_2)} \left[\ln\left(\frac{1-x}{r_2}\right) - i\pi\right]$$

with

$$4\pi B_{YFS}(s, m_e) = \left(4 \ln \frac{m_\gamma}{m_e} + 1\right)(L-1-i\pi) - L^2 - 1 + \frac{4\pi^2}{3}$$
$$+ i\pi(2L-1)$$

the YFS infrared factor,

$L = \ln\left(\frac{s}{m_e^2}\right)$ the "big logarithm",

$r_i = \frac{2p_i \cdot k}{s}$ the "collinear photon factors",

$x = r_1 + r_2 = \frac{E_\gamma}{E_{beam}}$ the fractional radiated energy,

$$R(x,y) = \ln^2(1-x) + 2\ln(1-x)\left[\ln\left(\frac{y}{1-x}\right) + i\pi\right] + 2\,Sp(x+y)$$
$$- 2\,Sp\left(\frac{y}{1-x}\right).$$

$$Sp(z) = -\int_0^z \frac{\ln(1-t)}{t}\,dt$$ is the Spence Di-logarithm function.

The Amplitude can be expressed in a leading log expansion, in terms of the powers of the "big logarithm" $L = \ln\left(\frac{s}{m_e^2}\right)$ obtained in the <u>integrated</u> cross section. ($L = 16$ at $\sqrt{s} = 200\,\text{GeV}$)

Leading log (LL) : Order L^2

Next-to-Leading log (NLL): Order L.

The terms through order NLL come from collinear limits where $r_1 \to 0$ or $r_2 \to 0$ (neglecting massen):

$$f_0^{pNLL} = 2(L-1-i\pi) + 2\ln(1-r_1)(\ln r_2 + i\pi) + 2\ln(1-r_2)(\ln r_1 + i\pi) - \ln^2(1-r_1-r_2) + 3\ln(1-r_1-r_2) + 2\,Sp(r_1+r_2) + \frac{r_1+r_2}{2(1-r_1-r_2)} + \frac{\sigma\lambda_1}{2}\,\frac{r_2-r_1}{1-r_1-r_2}.$$

Mass corrections were added following the approach of the CALCUL collaboration, which adds the mass terms most important in the collinear limits by the prescription

$$|\mathcal{M}^{(m)}_{1\gamma}|^2 = |\mathcal{M}^{(0)}_{1\gamma}|^2 - \frac{e^2 m^2}{q\cdot k} |\mathcal{M}^{(0)}_{Born}(q-k)|^2$$

for k radiated nearly parallel to q (fermion line) and $\mathcal{M}^{(0)}_{Born}$ without radiated photon.

The net effect is that f_0 is replaced by $f_0 + f_m$ with*

$$f_m = \frac{2m_e^2}{s}\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)\frac{z}{(1-r_1)^2 + (1-r_2)^2} *$$

$$* \left\{ f_0 + \frac{\alpha}{\pi}\left[\ln z\left(L + \frac{1}{2}(\ln z - 1)\right) - L - \ln z + 1\right]\right\}$$

*(To be precise, f_0 is the complete massless virtual photon correction factor, not just the terms displayed earlier)

The f_1, f_2 terms are complicated, but do not contain NLL contributions. They can be found in the Phys. Rev. D65 paper.

For cross-checks, this result has been compared to various other results:

1) Igarashi, Nakazawa, Nucl. Phys. B288, 301 (1987)
   - spin-averaged cross section, fully differential in photon variables τ_1, τ_2, no mass corrections

2) Berends, et al, Nucl. Phys. B297 (1988), 429
   - spin-averaged cross section, differential only in $x = \tau_1 + \tau_2$. Has mass corrections.

3) Kühn, Rodrigo, et al, Eur. Phys. J. C22, 81 (2001) and Eur. Phys. J. C25, 215 (2002)
   - spin averaged leptonic tensor, fully differential with mass corrections

The comparison 3 is a new one, which should be equivalent in all respects to our calculation, except that it was computed for a different purpose.

Rodrigo & Kuhn (et al) were interested in <u>radiative return</u>, a way to explore a wide range of energies in a single experiment

$$e^+(p_1) + e^-(p_2) \to \gamma(k) + \gamma^*(Q)$$

$\hookrightarrow$ Hadrons

CMS energies $\sqrt{Q^2} \sim 1.4\,\text{GeV} - 2\,\text{GeV}$ could be reached by radiating most of the original CMS energy into the photon. Small angle radiation is especially useful due to the rate enhancement.

Example: $\pi^+\pi^-$ final state investigated by DAΦNE.

The cross sections were implemented in a Monte Carlo Program <u>Pokhara</u>.

The calculation is done using a "Leptonic Tensor" for the initial state: For pure photon exchange

$$\overline{|\mathcal{M}_1^{ISR(i)}|^2} = \frac{1}{4ss'r_1r_2} L_i^{\mu\nu} H_{\mu\nu}$$

where $L^{\mu\nu}$ represents the initial state contribution, and has the general form

$$L_i^{\mu\nu} = a_{00}^{(i)} g^{\mu\nu} s + a_{11}^{(i)} p_1^\mu p_1^\nu + a_{22}^{(i)} p_2^\mu p_2^\nu$$
$$+ a_{12}^{(i)} (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) + i\pi a_{-1}^{(i)} (p_1^\mu p_2^\nu - p_2^\mu p_1^\nu)$$

For pure real photon emission (no virtual corrections),

$$a_{00}^{(0)} = -e^4((1-r_1)^2 + (1-r_2)^2)$$

$$a_{11}^{(0)} = a_{22}^{(0)} = -4e^4 z \qquad \text{with } z = 1-x = 1-r_1-r_2.$$

$$a_{12}^{(0)} = a_{-1}^{(0)} = 0$$

(neglecting masses)

$H^{\mu\nu} = J^\mu J^{\nu\dagger}$ in terms of the final state current J^μ. For $\pi^+\pi^-$ final state,

$$J_{2\pi}^\mu = ie F_{2\pi}(Q^2)(q_{\pi^+}^\mu - q_{\pi^-}^\mu).$$

But we would like to compare with $e^+e^- \to f\bar{f}$.

For $f\bar{f}$ final state, with $f(p_3, \lambda_3)$, $\bar{f}(p_4, \lambda_4)$,

$$J^\mu = ie\langle p_3 \lambda_3 | \gamma^\mu | p_4 \lambda_4 \rangle,$$

$$H_{\mu\nu} = J_\mu J_\nu^* = e^2 \, \mathrm{tr}(\not{p}_3 \gamma_\mu \not{p}_4 \gamma_\nu)$$

$$= 4e^2 (p_{3\mu} p_{4\nu} + p_{4\mu} p_{3\nu} - p_3 \cdot p_4 \, g_{\mu\nu}).$$

Contracting L with H gives (neglecting mass corrections)

$$|\mathcal{M}_1^{ISR(1)}|^2 = \{-4 a_{00} s s' + 2a_{11} t_1 u_1 + 2a_{22} t_2 u_2 + 2a_{12}(t_1 t_2 + u_1 u_2 - s's)\} \frac{e^2}{4 s s' r_1 r_2}$$

with $s = (p_1 + p_2)^2 = 4E_{beam}^2$, $s' = (p_3 + p_4)^2 = sz$,

$t_1 = (p_1 - p_3)^2$, $t_2 = (p_2 - p_4)^2$, $u_1 = (p_1 - p_4)^2$, $u_2 = (p_2 - p_3)^2$.

The tree-level result is found to be (as expected)

$$|\mathcal{M}_1^{ISR(0)}|^2 = \frac{e^6}{s^2 r_1 r_2} (t_1^2 + t_2^2 + u_1^2 + u_2^2).$$

The real + virtual squared amplitude can be written as a term proportional to the tree level result, plus extra terms:

$$|\mathcal{M}_1^{ISR(1)}|^2 = |\mathcal{M}_1^{ISR(0)}|^2 \left(\frac{a_{\infty}^{(1)}}{a_{\infty}^{(0)}} - \frac{r_1^2 + r_2^2}{2z} \frac{a_{12}^{(1)}}{a_{\infty}^{(0)}} \right)$$

$$+ \frac{2e^6}{s^2 r_1 r_2} \frac{t_1 u_1}{a_{\infty}^{(0)}} \left(a_{\infty}^{(1)} + \frac{1}{2} a_{12}^{(1)} - \frac{a_{\infty}^{(0)}}{a_{11}^{(0)}} a_{11}^{(1)} \right)$$

$$+ \frac{2e^6}{s^2 r_1 r_2} \frac{t_2 u_2}{a_{\infty}^{(0)}} \left(a_{\infty}^{(1)} + \frac{1}{2} a_{12}^{(1)} - \frac{a_{\infty}^{(0)}}{a_{22}^{(0)}} a_{22}^{(1)} \right)$$

$$- \frac{e^6}{s^2 r_1 r_2} \frac{a_{12}^{(1)}}{4z} \left[(r_3 - r_1)^2 + (r_3 - r_2)^2 \right]$$

with $r_3 = \frac{2 p_3 \cdot k}{s}$.

It is expected that the coefficient of $|\mathcal{M}_1^{ISR(0)}|^2$ should be essentially the same as $\mathrm{Re}\, f_0$ in our expression, averaged over helicities, and that the remaining terms should not contribute to NLL order.

The coefficient functions are

$$a_{00}^{(1)} = a_{00}^{(0)}\left[4\pi B_{gfs}(s,m_e) + 2(\ell-1) + 3\ln z\right]$$
$$- xz - 2r_1r_2 - 2\left(z + \frac{2r_1r_2}{z}\right)\ln z$$
$$+ r_1\left(4 - r_1 - \frac{3(1+z)}{1-r_2}\right)\ln(r_2/z)$$
$$+ r_2\left(4 - r_2 - \frac{3(1+z)}{1-r_1}\right)\ln(r_1/z)$$
$$-2\left[1 + (1-r_2)^2 + \frac{r_1}{r_2}z\right]L(r_1)$$
$$-2\left[1 + (1-r_1)^2 + \frac{r_2 z}{r_1}\right]L(r_2)$$

$$a_{11}^{(1)} = a_{11}^{(0)}\left[4\pi B_{gfs}(s,m_e) + 2(\ell-1) + 3\ln z\right]$$
$$+2(1+z^2)^2\left(\frac{1}{1-r_1} - \frac{1}{x}\right) - \frac{8r_1(1-r_2)}{x} - 2z\left(1+\frac{2}{r_2}\right)\ln\left(\frac{r_1}{z}\right)$$
$$-\frac{4z}{x}\left[(1-r_2)\left(\frac{1}{r_2} + \frac{z}{r_1} + \frac{2r_1}{x}\right) + \frac{2z}{x}\right]\ln z$$
$$-2z\left[\frac{(2-3r_1)(1-r_2)^2}{r_1(1-r_1)^2}\right]\ln\left(\frac{r_2}{z}\right) - 4z\left(1+\frac{1}{r_2^2}\right)L(r_1)$$
$$-4z\left(3 + \frac{2z}{r_1} + \frac{z^2}{r_1^2}\right)L(r_2)$$

$$a_{22}^{(1)} = a_{11}^{(1)}(r_1 \leftrightarrow r_2)$$

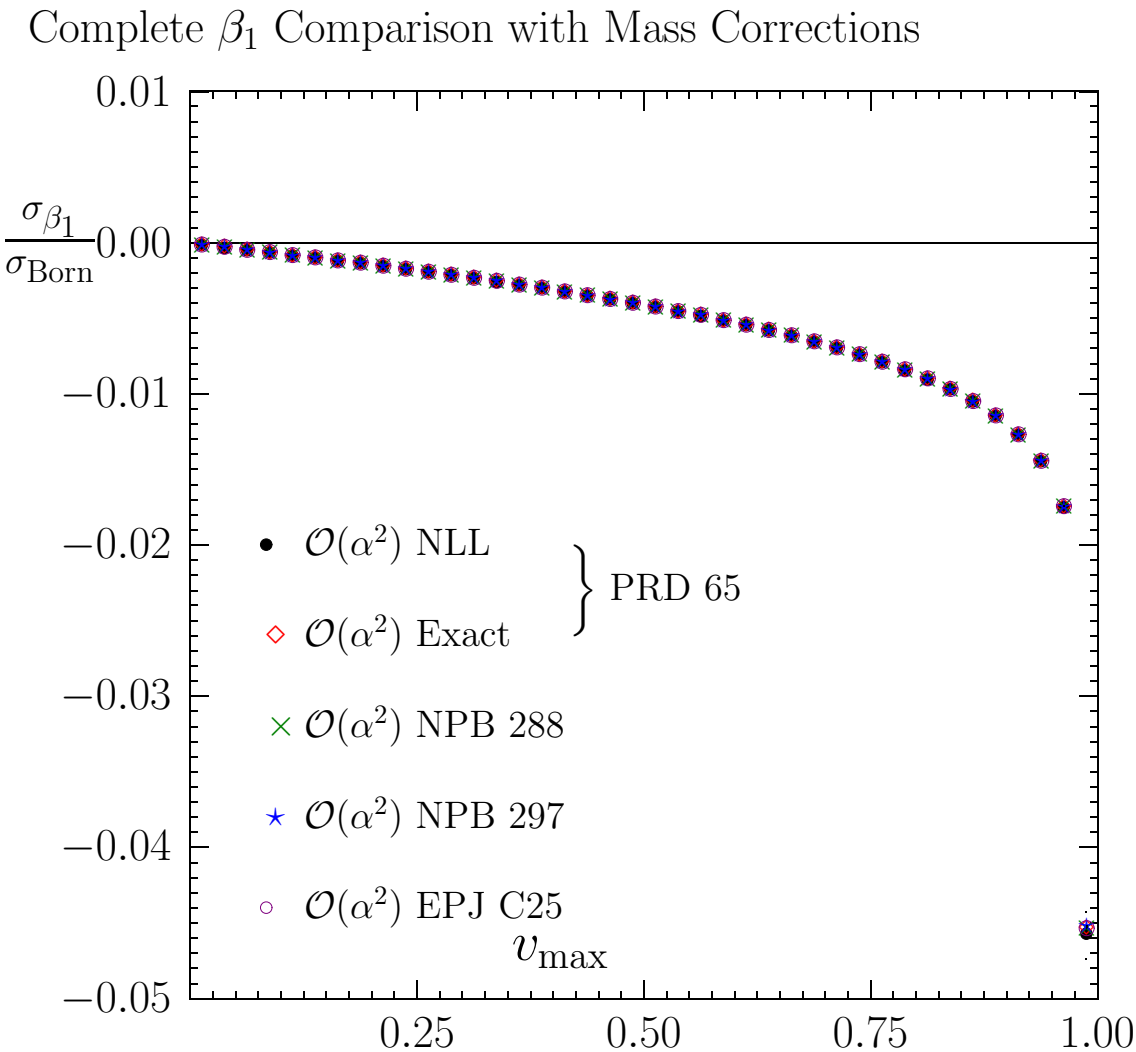
$$a_{12}^{(1)} = \frac{2z}{1-r_1} - \frac{4z + (r_1-r_2)^2}{x} - z\left(\frac{z}{r_1r_2} + \frac{1+z-2r_1r_2}{x^2}\right)\ln z$$
$$-\frac{4z}{1-r_2}\left(1 - r_1 + \frac{z}{r_2} - \frac{z}{2(1-r_1)}\right)\ln\left(\frac{r_2}{z}\right)$$
$$-4z\left(1 + \frac{z}{r_2} + \frac{z}{r_2^2}\right)L(r_1)$$
$$+ (r_1 \leftrightarrow r_2)$$

The $\frac{1}{r_i}$ and $\frac{1}{x}$ factors are apparent sources of singularities in the collinear & soft limits. They all turn out to be compensated in the numerators, but only after a 2nd order expansion.

$$L(y_i) = Sp\left(-\frac{y_i}{z}\right) - Sp\left(1-\frac{1}{z}\right) + \ln(z+y_i)\ln\left(\frac{y_i}{z}\right)$$

- The stabilized functions were programmed in the KK monte-carlo generator and compared to our expressions. Agreement on the order of 10^{-6} of the Born cross section is found, for much of the kinematic range, up to $\sim 10^{-5}$ at large V_{max}.

- Analytic agreement is found between a_{00} and the helicity averaged version of f_0 to NLL order.



Comparisons of the YFS residual β_1 for the complete cross section $e^+e^- \to f\overline{f} + \gamma$, normalized with respect to the Born cross section, as a function of the cut $v_{\rm max}$ on the maximum energy radiated into the photon relative to the beam energy. The functions are compared in the KK Monte Carlo for 100M events.

PRD 65 S. Jadach, M. Melles, B.F.L. Ward and S.A. Yost, *Phys. Rev.* **D65** (2002) 73030.

EPJ C25 J.H. Kühn and G. Rodrigo, *Eur. Phys. J.* **C25** (2002) 215.

NPB 297 F.A. Berends, W.L. Van Neerven and G.J.H. Burgers, *Nucl. Phys.* **B297** (1988) 429.

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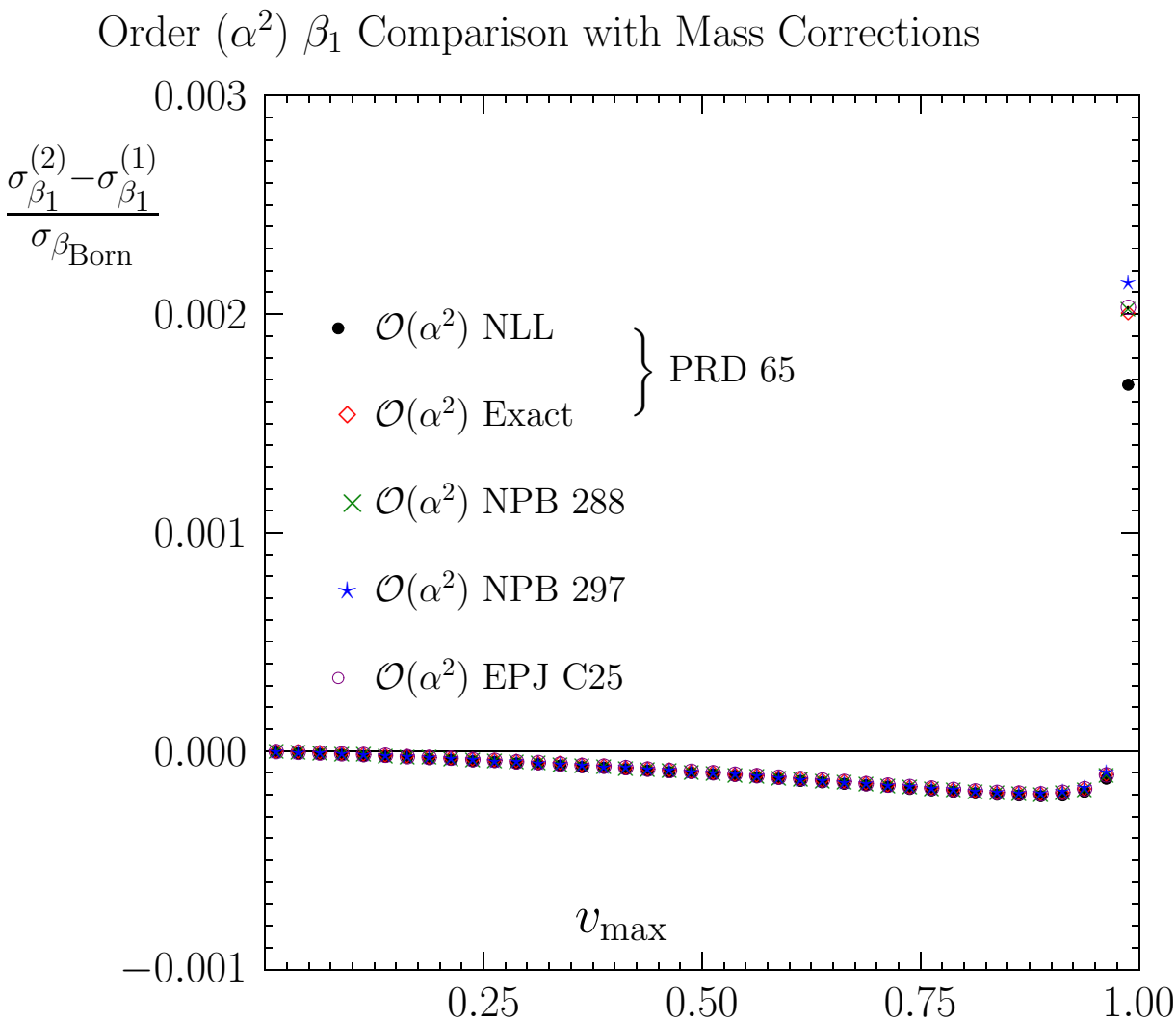

The order (α^2) virtual correction to β_1 as a function of the cut $v_{\rm max}$ on the maximum fraction of the beam energy radiated into the emitted photon. The functions are compared in the KK Monte Carlo for 100M events.

PRD 65 S. Jadach, M. Melles, B.F.L. Ward and S.A. Yost, *Phys. Rev.* **D65** (2002) 73030.

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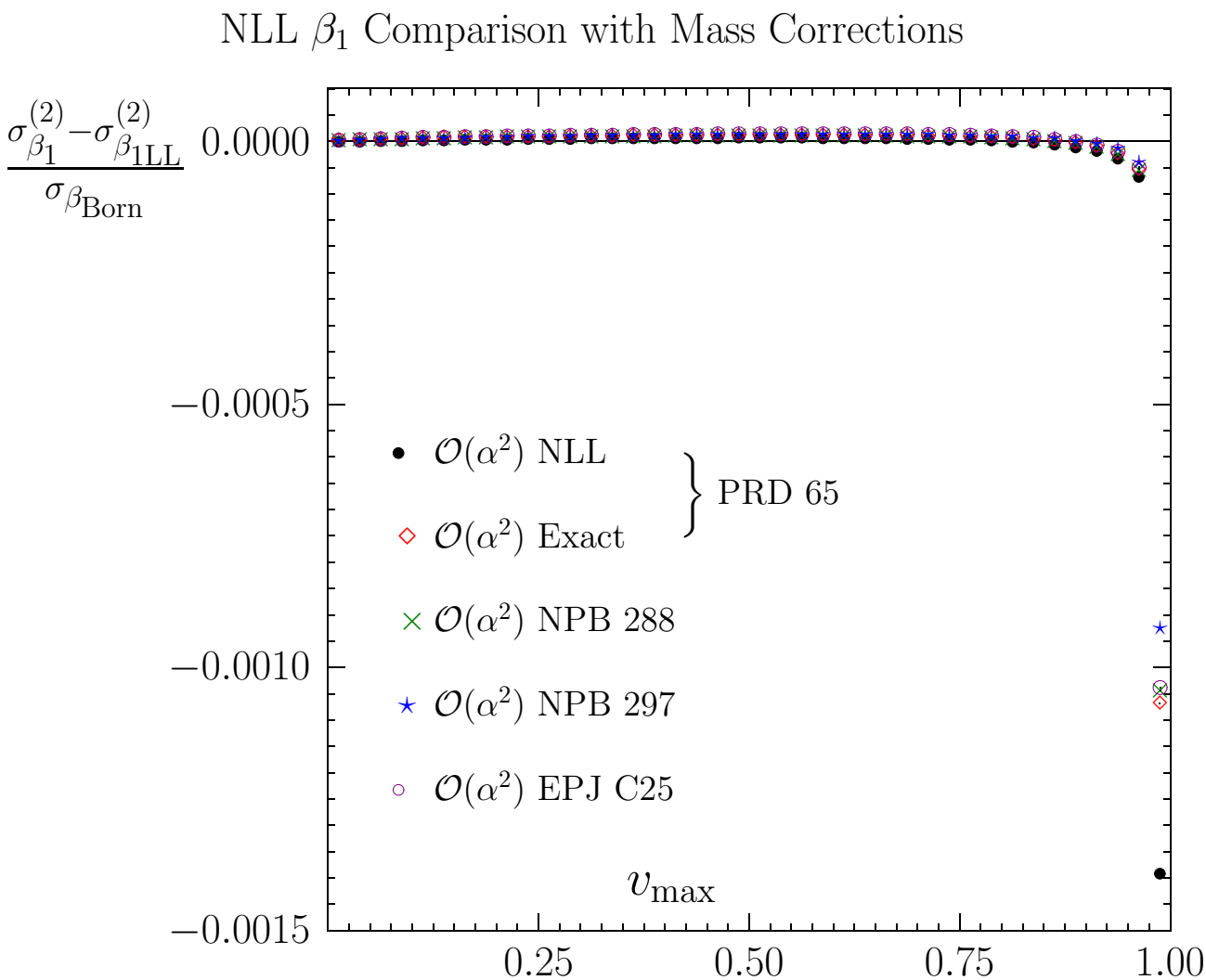

The residual part of the cross sections after subtracting common "LL" (order $(\alpha^2 L^2)$) terms is compared for four different expressions, for cut $v_{\max}$ on the maximum energy fraction carried by the emitted photon. The last data point is off-scale on this graph. The functions are compared in the KK Monte Carlo for 100M events.

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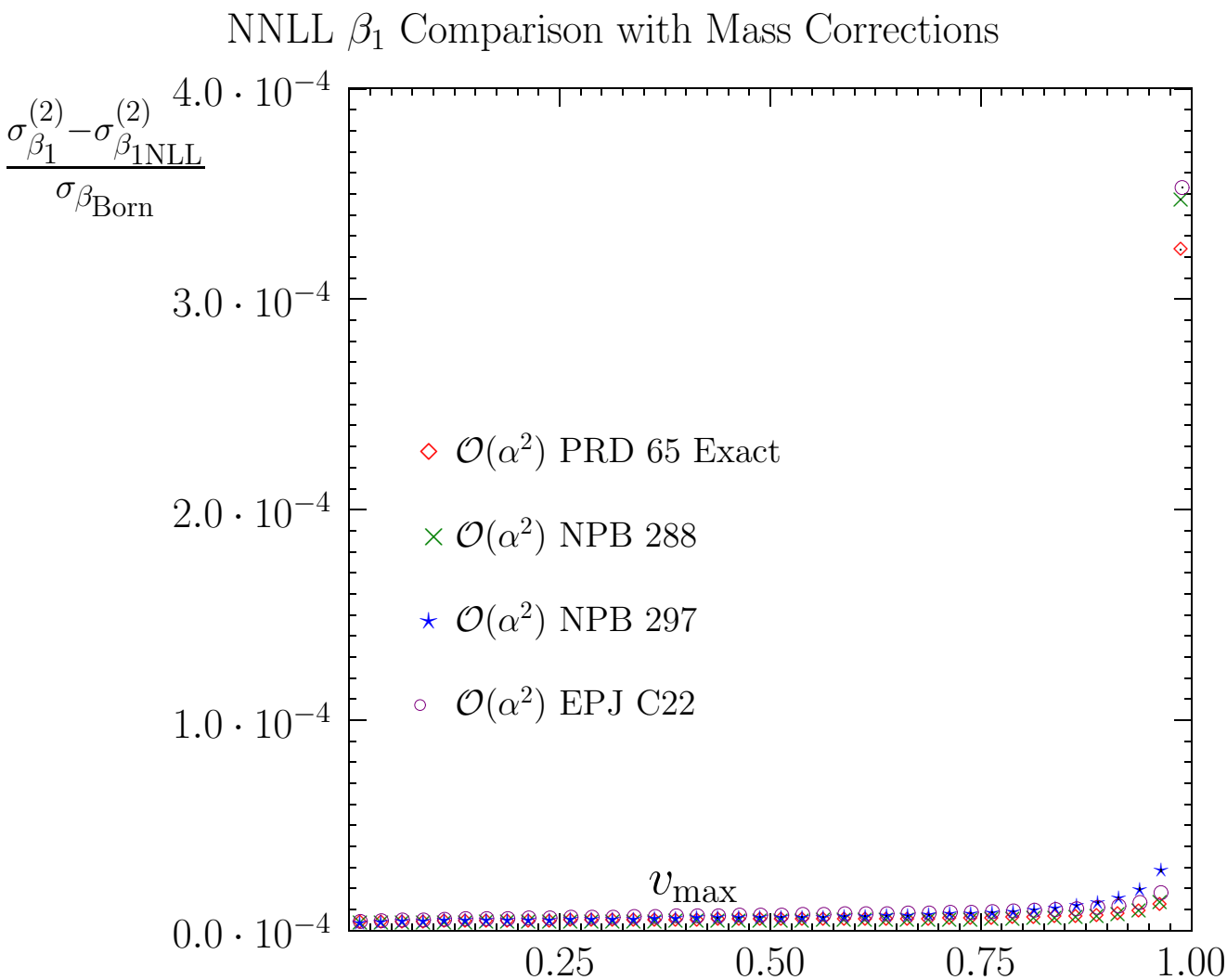

The residual part of the cross sections after subtracting common "NLL" (order $(\alpha^2 L)$) terms is compared for four different expressions, for cut v_{max} on the maximum energy fraction carried by the emitted photon.

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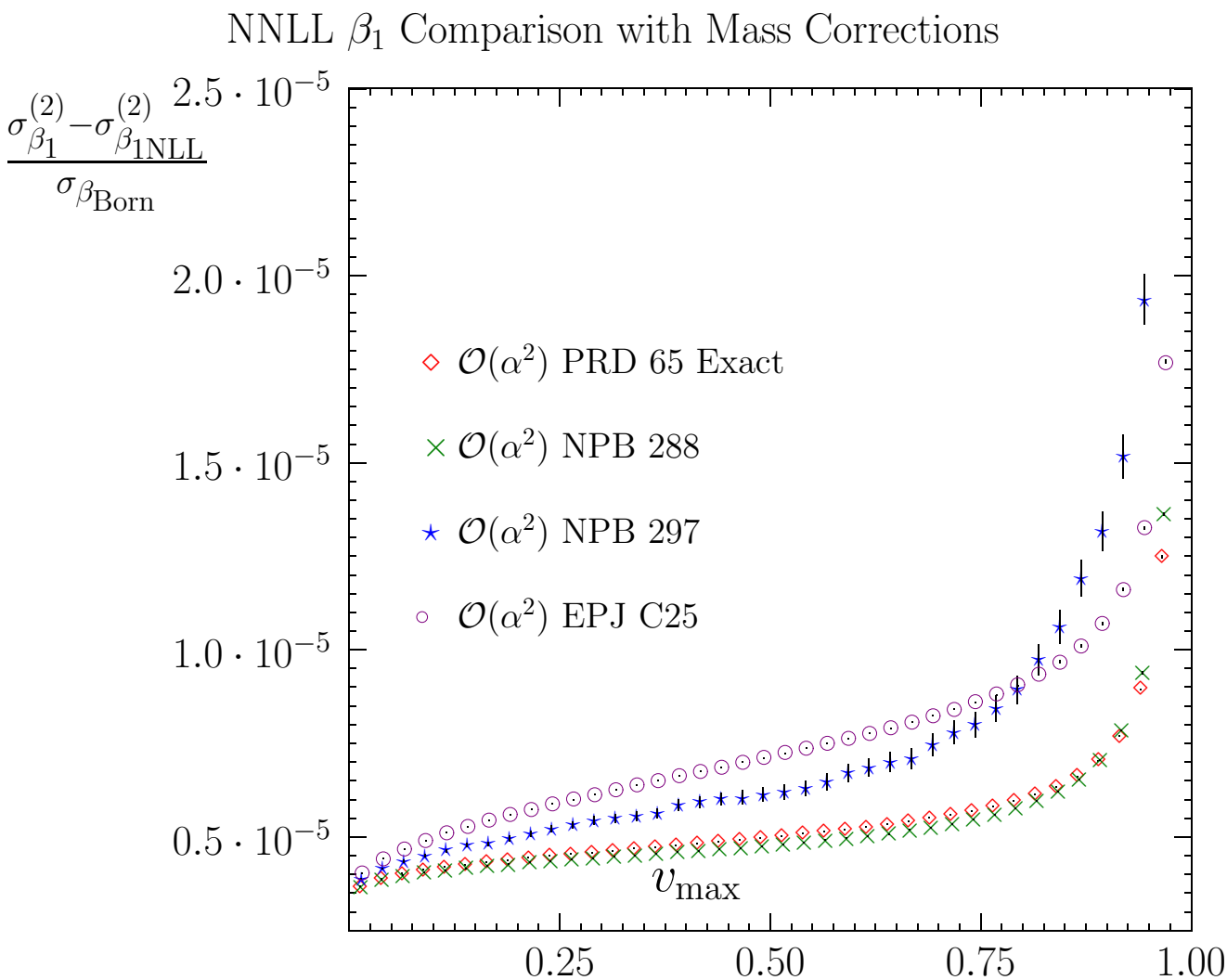

Closer view of the residual part of the cross sections after subtracting common "NLL" (order $(\alpha^2 L)$) terms is compared for four different expressions, for cut v_{max} on the maximum energy fraction carried by the emitted photon. The last data point is off-scale on this graph.

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