# **Theory of Parton Distributions**

### **Sven-Olaf Moch**

**DESY** Zeuthen

- 1. The Roadmap to Precision
- 2. The Three-loop Splitting Functions\*
- 3. The Summary

\* in collaboration with J.A.M. Vermaseren and A. Vogt, hep-ph/0403192, hep-ph/0404111

- Workshop HERA and the LHC, DESY Hamburg, June 3, 2004 -

### The Roadmap to Precision

### Structure of the proton

- Structure functions  $F_2, F_3, F_L$  in deep-inelastic scattering
  - scaling violations —> precision test of perturbative QCD
- Parton distributions
  - gluon distribution at small *x*, quark valence and sea distribution
  - important input for hard scattering reactions at hadron colliders
    - $\longrightarrow$  precise parton luminosity at LHC for Higgs or SUSY searches

#### How well do we know (un)-polarized parton distributions?

## $\alpha_{\it s}$ from DIS

- Fundamental parameter of Standard Model
  - determination in inclusive DIS independent of hadronic final state  $\longrightarrow$  ideal case

How well do we know  $\alpha_s$ ?

### **PDFs from HERA to LHC**





- LHC parton luminosity



### Recent determinations of $\alpha_s$

Bethke hep-ex/0211012

- NLO QCD analysis of HERA data for  $F_2(x, Q^2)$ H1 coll. hep-ph/0012053

$$\alpha_s(M_Z^2) = 0.115 \pm 0.002(\exp) \pm 0.005(\text{theo})$$

### **Future**

(

- NNLO QCD analysis of HERA data for  $F_2(x, Q^2)$ in 2006

$$\alpha_s(M_Z^2) = x \pm 0.001(\exp) \pm 0.001(\text{theo})$$

### **PDF uncertainties**

- QCD analyses require many choices, should be reflected in PDF uncertainties Böttcher
- Allowed functional form of PDF  $xf(x,Q_0^2) = Ax^b(1-x)^c(1+dx+...)$
- Treatment of heavy quarks

. . .

- Fixed flavor number scheme
  - $\longrightarrow n_f$  light flavors + heavy quark of mass *m* at low scales
  - $\longrightarrow n_f + 1$  light flavors at high scales
- Variable flavor number schemes
  - $\longrightarrow$  matching of two distinct theories through NNLO

Aivazis, Collins, Olness, Tung '94; Thorne, Roberts '98; Chuvakin, Smith, van Neerven '00

- Scale dependence  $\longrightarrow$  variation of renormalization / factorization scale
  - gluons : stat.  $\oplus$  syst.  $\simeq$  input  $\simeq$  scale error ; quarks : scale error already dominates
  - NNLO improvement of theory needed —> three-loop splitting functions

### **The Three-loop Splitting Functions**

### **Evolution of parton distribution**

– Non-singlet and singlet distributions  $q^{\pm}$ ,  $q^{\scriptscriptstyle 
m V}$  and  $q_{\scriptscriptstyle 
m S}$ , g

 $\begin{array}{lll} q_{\mathrm{ns},ik}^{\pm} &=& q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) & \text{flavour asymmetries} \\ \\ q_{\mathrm{ns}}^{\mathrm{v}} &=& \sum_{r=1}^{n_f} (q_r - \bar{q}_r) & \text{total valence distribution} \\ \\ q_{\mathrm{s}} &=& \sum_{r=1}^{n_f} (q_r + \bar{q}_r) & \text{flavour singlet distribution,} & f_{\mathrm{s}} = \left( \begin{array}{c} q_{\mathrm{s}} \\ g \end{array} \right) \end{array}$ 

- Splitting function combinations

$$\begin{split} P_{\rm ns}^{\pm}, & P_{\rm ns}^{\rm v} = P_{\rm ns}^{-} + P_{\rm ns}^{\rm s} & \text{non-singlet} \\ P_{\rm s} = \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}, & P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps} & \text{singlet} \end{split}$$

- Evolution equations :  $2n_f - 1$  scalar non-singlet equations and  $2 \times 2$  singlet equations  $\frac{d}{d \ln \mu_f^2} f(x, \mu_f^2) = \left[ P(\alpha_s(\mu_f^2)) \otimes f(\mu_f^2) \right](x)$ 

### The calculation (in a nut shell)

- Calculate anomalous dimensions (Mellin moments of splitting functions)
  - $\longrightarrow$  divergence of Feynman diagrams in dimensional regularization  $D = 4 2\epsilon$

$$\gamma_{\rm ij}^{(n)}(N) = -\int_0^1 dx \, x^{N-1} P_{\rm ij}^{(n)}(x)$$

- One-loop Feynman diagrams  $\longrightarrow$  in total 18 for  $\gamma_{ij}^{(0)}$  /  $P_{ij}^{(0)}$ (pencil + paper)
- $\begin{array}{l} \mbox{ Two-loop Feynman diagrams} \\ \longrightarrow \mbox{ in total 350 for } \gamma^{(1)}_{ij} \mbox{ / } P^{(1)}_{ij} \\ \mbox{ (simple computer algebra)} \end{array}$
- Three-loop Feynman diagrams  $\rightarrow$  in total 9607 for  $\gamma_{ij}^{(2)} / P_{ij}^{(2)}$ (cutting edge technology  $\rightarrow$  computer algebra system FORM Vermaseren '89-'04)



### Mathematical intermezzo

- Mellin *N*-space : harmonic sums  $S_{m_1,...,m_k}(N)$ Gonzalez-Arroyo, Lopez, Ynduráin '79 ; Vermaseren '98 ; Blümlein, Kurth '98
  - recursive definition  $S_{m_1,\ldots,m_k}(N) = \sum_{i=1}^N \frac{1}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$
  - algebra of multiplication  $S_j(N)S_k(N) \longrightarrow S_{\{j,k\}}(N)$
- Bjorken *x*-space : harmonic polylogarithms  $H_{m_1,...,m_k}(x)$ Goncharov '98 ; Borwein, Bradley, Broadhurst, Lisonek '99 ; Remiddi, Vermaseren '99
  - Basic functions of lowest weight

$$H_0(x) = \ln x$$
,  $H_1(x) = -\ln(1-x)$ ,  $H_{-1}(x) = \ln(1+x)$ 

Higher functions defined by recursion

$$H_{m_1,...,m_w}(x) = \int_0^x dz \ f_{m_1}(z) \ H_{m_2,...,m_w}(z)$$

$$f_0(x) = \frac{1}{x}, \qquad f_1(x) = \frac{1}{1-x}, \qquad f_{-1}(x) = \frac{1}{1+x}$$

- Inverse Mellin transformation of harmonic sums  $\longrightarrow$  harmonic polylogarithms in x space
- Unique mapping :

$$H_{m_1,\ldots,m_w}(x)/(1\pm x)\longleftrightarrow S_{n_1,\ldots,n_{w+1}}(N)$$

# LO and NLO singlet splitting functions

$$\begin{aligned} P_{\rm ps}^{(0)}(x) &= 0 \\ P_{\rm qg}^{(0)}(x) &= 2 n_f p_{\rm qg}(x) \\ P_{\rm gq}^{(0)}(x) &= 2 C_F p_{\rm gq}(x) \\ P_{\rm gg}^{(0)}(x) &= C_A \left(4 p_{\rm gg}(x) + \frac{11}{3} \delta(1-x)\right) - \frac{2}{3} n_f \delta(1-x) \end{aligned}$$

$$\begin{split} P_{\rm ps}^{(1)}(x) &= 4C_F n_f \Big( \frac{20}{9} \frac{1}{x} - 2 + 6x - 4{\rm H}_0 + x^2 \Big[ \frac{8}{3} {\rm H}_0 - \frac{56}{9} \Big] + (1+x) \Big[ 5{\rm H}_0 - 2{\rm H}_{0,0} \Big] \Big) \\ P_{\rm qg}^{(1)}(x) &= 4C_A n_f \Big( \frac{20}{9} \frac{1}{x} - 2 + 25x - 2p_{\rm qg}(-x){\rm H}_{-1,0} - 2p_{\rm qg}(x){\rm H}_{1,1} + x^2 \Big[ \frac{44}{3} {\rm H}_0 - \frac{218}{9} \Big] \\ &+ 4(1-x) \Big[ {\rm H}_{0,0} - 2{\rm H}_0 + x{\rm H}_1 \Big] - 4\zeta_2 x - 6{\rm H}_{0,0} + 9{\rm H}_0 \Big) + 4C_F n_f \Big( 2p_{\rm qg}(x) \Big[ {\rm H}_{1,0} + {\rm H}_{1,1} + {\rm H}_2 \\ &- \zeta_2 \Big] + 4x^2 \Big[ {\rm H}_0 + {\rm H}_{0,0} + \frac{5}{2} \Big] + 2(1-x) \Big[ {\rm H}_0 + {\rm H}_{0,0} - 2x{\rm H}_1 + \frac{29}{4} \Big] - \frac{15}{2} - {\rm H}_{0,0} - \frac{1}{2}{\rm H}_0 \Big) \\ P_{\rm gq}^{(1)}(x) &= 4C_A C_F \Big( \frac{1}{x} + 2p_{\rm gq}(x) \Big[ {\rm H}_{1,0} + {\rm H}_{1,1} + {\rm H}_2 - \frac{11}{6}{\rm H}_1 \Big] - x^2 \Big[ \frac{8}{3} {\rm H}_0 - \frac{44}{9} \Big] + 4\zeta_2 - 2 \\ &- 7{\rm H}_0 + 2{\rm H}_{0,0} - 2{\rm H}_1 x + (1+x) \Big[ 2{\rm H}_{0,0} - 5{\rm H}_0 + \frac{37}{9} \Big] - 2p_{\rm gq}(-x){\rm H}_{-1,0} \Big) - 4C_F n_f \Big( \frac{2}{3} x \\ &- p_{\rm gq}(x) \Big[ \frac{2}{3} {\rm H}_1 - \frac{10}{9} \Big] \Big) + 4C_F^2 \Big( p_{\rm gq}(x) \Big[ 3{\rm H}_1 - 2{\rm H}_{1,1} \Big] + (1+x) \Big[ {\rm H}_{0,0} - \frac{7}{2} + \frac{7}{2}{\rm H}_0 \Big] - 3{\rm H}_{0,0} \\ &+ 1 - \frac{3}{2} {\rm H}_0 + 2{\rm H}_1 x \Big) \\ \\ P_{\rm gg}^{(1)}(x) &= 4C_A n_f \Big( 1 - x - \frac{10}{9} p_{\rm gg}(x) - \frac{13}{9} \Big( \frac{1}{x} - x^2 \Big) - \frac{2}{3} (1+x){\rm H}_0 - \frac{2}{3} \delta(1-x) \Big) + 4C_A^2 \Big( 27 \\ &+ (1+x) \Big[ \frac{11}{3} {\rm H}_0 + 8{\rm H}_{0,0} - \frac{27}{2} \Big] + 2p_{\rm gg}(-x) \Big[ {\rm H}_{0,0} - 2{\rm H}_{-1,0} - \zeta_2 \Big] - \frac{67}{9} \Big( \frac{1}{x} - x^2 \Big) - 12{\rm H}_0 \\ &- \frac{44}{3} x^2 {\rm H}_0 + 2p_{\rm gg}(x) \Big[ \frac{67}{18} - \zeta_2 + {\rm H}_{0,0} + 2{\rm H}_{1,0} + 2{\rm H}_2 \Big] + \delta(1-x) \Big[ \frac{8}{3} + 3\zeta_3 \Big] \Big) + 4C_F n_f \Big( 2{\rm H}_0 \\ &+ \frac{2}{3} \frac{1}{x} + \frac{10}{3} x^2 - 12 + (1+x) \Big[ 4 - 5{\rm H}_0 - 2{\rm H}_{0,0} \Big] - \frac{1}{2} \delta(1-x) \Big) . \end{split}$$

#### NNLO singlet splitting functions

$$\begin{split} \mu_{1}^{(0)}(u) &= 16c_{1}c_{1}u_{1}\left(\frac{1}{4}+z^{2}\right)\left[\frac{13}{2}\pi L_{1,0}-\frac{14}{2}H_{0}+\frac{1}{2}H_{1,-2}\left[\frac{1}{2}+L_{1,-1,0}-2\pi L_{1,0,0}\right] \\ &-H_{1,-1}\right]+\frac{2}{3}c_{1}^{2}-z^{2}\right)\left[\frac{16}{2}c_{2}+H_{2,1}+s_{3}^{2}+\frac{2}{4}H_{0}-\frac{2761}{2}+\frac{572}{2}H_{1}+\frac{10}{3}H_{2}+H_{1}c_{2}-\frac{1}{6}H_{1,1}\right] \\ &-H_{1,0,0}+2H_{1,1,0}+2H_{1,1,1}\right]+(1-z)\left[\frac{128}{2}H_{1}+\frac{183}{3}+\frac{397}{2}H_{0,0}-\frac{13}{2}H_{2}+2H_{1}c_{2}-\frac{1}{6}H_{1,0}\right] \\ &-\frac{13}{4}H_{1,1}+2H_{1,1,0}+H_{1,1,1}\right]+(1-z)\left[\frac{128}{12}H_{2}+\frac{13}{3}+\frac{397}{2}H_{0,0}-\frac{13}{2}H_{1}+\frac{113}{2}c_{2}^{2}+2H_{1}c_{2}+\frac{1}{2}H_{1,0}+\frac{2}{2}H_{2,0}+\frac{1}{2}H_{1,1}+\frac{11}{2}H_{1}+\frac{113}{2}c_{2}^{2}+2H_{2}c_{2}+\frac{1}{2}H_{1,0}+\frac{1}{2}H_{2}+\frac$$

$$\begin{split} p^{(2)}_{\mu\nu}(x) &= 16C_{\mu}C_{\mu\nu}/[\rho_{eq}(x)]\frac{32}{3}H_{15}(x) - 4H_{1,11} + 3H_{2,0,0} - \frac{15}{12}H_{1,2} + \frac{9}{2}H_{1,1,0} + 3H_{2,1,0} \\ + H_{05}(x) - 2H_{2,1,1} + 4H_{15}(x) - \frac{173}{12}H_{15}(x) - \frac{551}{7}H_{0,0} + \frac{44}{9}_{5,1}(x) - \frac{49}{18}H_{1,2} - \frac{34}{9}H_{1,2} - \frac{34}{9}H_{1,2} - \frac{34}{12}H_{1,2} - \frac{11}{12}H_{1,2} + \frac{39}{12}H_{1,2} - \frac{31}{12}H_{1,2} - \frac{11}{12}H_{1,2} + \frac{39}{12}H_{1,2} + \frac{5}{12}H_{1,2} + \frac{39}{12}H_{1,2} + \frac{173}{12}H_{1,2} - \frac{123}{122} + \frac{233}{124}H_{1,1,1} \\ + 6H_{2,1} + 3H_{1,2,0} + 9H_{1,0}(x) + 6H_{1,1}(x) + H_{1,1,0} - 3H_{1,1,1} - 3H_{1,2} - 2H_{2,2} - 2H_{2,2} \end{split}$$

 $-2H_{3,0}-\frac{13}{2}H_0\zeta_2-13H_{-3,0}-\frac{13}{2}H_{3,1}+\frac{15}{2}H_3-\frac{2005}{64}+\frac{157}{4}\zeta_2+8\zeta_3+\frac{1291}{432}H_1+\frac{55}{12}H_{1,1}$  $+\frac{3}{3}H_{2}+\frac{1}{3}H_{2,1}+\frac{27}{4}H_{-1,0}-\frac{11}{2}H_{1,0,0}-8H_{2,0,0}-4\zeta_{2}^{2}+\frac{3}{2}H_{1,2}-H_{2,2}+\frac{5}{2}H_{1}\zeta_{2}+8H_{-1,-1,0}$  $+4H_{2,0}+\frac{3}{2}H_{2,1,1}-H_{-1}\zeta_2+7H_2\zeta_2+6H_{-2}\zeta_2+12H_{-2,-1,0}-6H_{-2,0,0}+x\Big[3H_{1,1,1}-H_{0,0}\zeta_2+12H_{-2,-1,0}-6H_{-2,0,0}+x\Big]$  $+\frac{9}{2}H_{-1,0,0}-\frac{35}{8}H_{1,0}+2H_4+3H_{1,1,0}+H_{-1,2}]+16C_A^2C_F(x^2[\frac{2}{2}H_1\zeta_2-\frac{2105}{91}-\frac{77}{19}H_{0,0})$  $-6H_3 + \frac{16}{3}\zeta_3 - 10H_{-1,0} - \frac{14}{3}H_{2,0} - \frac{2}{3}H_{-1}\zeta_2 - \frac{14}{3}H_{0,0,0} + \frac{104}{6}H_2 - \frac{4}{3}H_{1,0,0} + \frac{37}{6}H_{1,1}$  $+\frac{4}{3}H_{-1,-1,0}-\frac{104}{9}\zeta_{2}-\frac{8}{3}H_{2,1}+\frac{145}{18}H_{1,0}+\frac{4}{3}H_{-1,2}+\frac{2}{3}H_{1,1,1}-\frac{109}{27}H_{1}+\frac{8}{3}H_{-1,0,0}+6H_{0}\zeta_{2}$  $+4H_{3,1} - \frac{43}{6}H_{1,1,1} - \frac{10}{12}\zeta_2 - \frac{17}{3}H_{2,1} - \frac{71}{24}H_{1,0} - \frac{11}{6}H_{-2,0} - \frac{21}{2}\zeta_3 + \frac{3}{2}H_{1,0,0,0} - H_{1,-2,0}$  $+\frac{395}{54}H_0-2H_{1,0}\zeta_2-H_{1,1}\zeta_2-\frac{55}{12}H_{1,1,0}+2H_{1,1,0,0}+4H_{1,1,1,0}+2H_{1,1,1,1}+4H_{1,1,2}-\frac{55}{12}H_1$ 54 12 12 + $6H_{1,2,0} + 4H_{1,2,1} + 4H_{1,3} + 3H_{2,1,0} + 3H_{2,2}$  +  $p_{gg}(-x)\left[\frac{23}{2}H_{-1}\zeta_3 + 5H_{-2}\zeta_2 + 2H_{-2,-1,0}\right]$  $+\frac{109}{12}H_{-1,0} + H_0\zeta_3 + \frac{17}{5}\zeta_2^2 + \frac{1}{7}H_1\zeta_2 + 2H_2\zeta_2 - \frac{65}{7}H_{1,1} - \frac{19}{7}H_{-1,-1,0} - 4H_{3,0} - 3H_{2,0,0}$  $+\frac{11}{6}H_0\zeta_2\Big]+(1-x)\Big[\frac{41699}{2592}-3H_{-2,-1,0}-\frac{3}{2}H_{-2}\zeta_2-\frac{128}{9}\zeta_2-4H_{3,0}+\frac{21}{3}\zeta_3-\frac{5}{2}H_{-2,0,0}$  $\begin{array}{c} 6 \\ -7 \\ -7 \\ H_1 \\ \zeta_2 + \frac{97}{12} \\ H_{1,0,0} + \frac{10}{2} \\ H_{1,0,0} + \frac{245}{12} \\ H_3 - 8 \\ H_{0,0,0,0} \\ + (1+x) \\ \left[ 4 \\ H_{3,1} - \\ H_{2,1,1} + \frac{29}{2} \\ H_{-1,2} \\ \end{array} \right]$  $+\frac{17}{6}H_{-2,0} - \frac{12}{12}H_{2,0} - \frac{31}{12}H_{2,1} + \frac{1}{2}H_{2,0,0} - H_2\zeta_2 + \frac{61}{26}H_{1,0} - 4H_0\zeta_3 - \frac{13}{2}H_{-1}\zeta_2 - \frac{46}{26}H_{-1,-1,0}$  $+\frac{25}{4}H_4+\frac{93}{4}H_0\zeta_2-\frac{55}{59}H_{1,1}-\frac{71}{18}H_2+\frac{49}{18}H_{0,0}-\frac{13}{2}H_{0,0}\zeta_2-\frac{40}{47}\zeta_2^2\Big]+\frac{613}{2592}-\frac{31}{2}H_{-2}\zeta_2$  $-\frac{67}{40}\zeta_{2}^{2}+\frac{29}{6}H_{-1,2}-H_{-1,0}+8H_{-2,2}+25H_{0}\zeta_{2}+\frac{412}{9}H_{1}+\frac{928}{9}H_{0}+\frac{1}{4}H_{4}-65H_{3}-38H_{0,0}$  $-9H_{-3,0} - \frac{17}{3}H_{0,0,0} + x \left[\frac{27}{2}H_{-1,0} - \frac{1}{2}H_{0,0,0,0} + \frac{3}{4}H_{0,0}\xi_2 + \frac{1}{2}H_{-3,0} - 14H_{0,0,0} + \frac{1}{12}H_{1,1,1}\right]$  $-\frac{43}{36}\zeta_2 - \frac{1}{2}H_2\zeta_2 + \frac{7}{72}H_0 + \frac{749}{54}H_1 + \frac{135}{4}\zeta_3 + \frac{97}{24}H_{1,0} + \frac{43}{12}H_1\zeta_2 - \frac{85}{12}H_{-1}\zeta_2 - \frac{13}{3}H_{1,0,0}$  $\frac{53}{12}H_2 + \frac{39}{4}H_{1,1} - 2H_{3,1} + \frac{13}{6}H_{-1,-1,0} + \frac{7}{4}H_{2,0,0} - 4H_{1,1,0} - 4H_{1,2} + 16C_F n_f^2 \left(\frac{1}{9} - \frac{11}{9}\right)$  $+\frac{2}{6}x - \frac{1}{6}xH_1 + \frac{1}{6}p_{B_1}(x)\left[H_{1,1} - \frac{5}{3}H_1\right] + 16C_F^2n_f\left(\frac{4}{6}x^2\left[H_{0,0} - \frac{11}{6}H_0 - \frac{7}{2} + H_{-1,0}\right]\right]$ 

 $\begin{array}{l} -6H_{1,3}+\frac{49}{4}\zeta_2\right]+p_{48}(-x)\left[\frac{17}{2}H_{-1}\zeta_3-\frac{5}{2}H_{-1,-1,0}-\frac{5}{2}H_{-1,2}-\frac{9}{2}H_{-1,0}+\frac{5}{2}H_{-2,0}+\frac{3}{2}H_{-1,0,0}\right]\\ -2H_{3,1}-2H_{4,1}-6H_{-2,2,1}-6H_{-2,0,1}-6H_{-2,0,0}+2H_{0,0}\zeta_2+9H_{-2}\zeta_2+3H_{-1,-2,0}-2H_{-1,2,1}\right]\end{array}$  $-6H_{-1,-1,-1,0}+6H_{-1,-1,0,0}+6H_{-1,-1,2}+9H_{-1,0}\zeta_2-9H_{-1,-1}\zeta_2-2H_{-1,2,0}-\frac{11}{2}H_{-1,0,0,0}$  $-6H_{-1,3}\Big] + (\frac{1}{x} - x^2) \Big[\frac{55}{12} - 4\zeta_3 + \frac{23}{9}H_{1,0} - \frac{4}{3}H_{1,1,0}\Big] + (\frac{1}{x} + x^2) \Big[\frac{2}{3}H_{1,0,0} - \frac{371}{109}H_1 + \frac{23}{9}H_{1,1}\Big]$  $-\frac{2}{2}H_{1,1,1} + (1-x) \left[ 6H_{2,1,0} + 3H_{2,1,1} - \frac{5}{4}H_{1,1,1} - 7H_{2,0,0} - 2H_{1,2} + 39H_0\zeta_3 - 4H_2\zeta_2 - \frac{16}{2}\zeta_3 - \frac{16}{2$  $\begin{array}{c} 3^{-6,11} & (-6)$ +(1+x) $H_{-1,0,0} - 10H_{-2}\zeta_2 + 6H_{-2,0,0} + 2H_{0,0}\zeta_2 - 9H_{-1,-1,0} - 7H_{-1,2} - 9H_{-2,0} - 2H_3$ ,  $-4H_{-2-1,0} - 4H_4 - 4H_{3,0} - 4H_{0,0,0,0} + \frac{37}{2}H_{-1,0} + \frac{5}{2}(1+x)H_{-1}\zeta_2 - 4H_{-2,0,0} + 2H_{0,0}\zeta_2$  $+H_{2}\zeta_{2}-3H_{1,1,0}+2H_{0,0,0,0}+H_{-3,0}-9H_{2,1,0}-\frac{9}{5}H_{2,1,1}+\frac{11}{2}H_{1,1,1}+\frac{19}{2}H_{2,0,0}+\frac{9}{5}H_{1,2}$  $-\frac{91}{2}H_{0}\zeta_{3}+8H_{-2}\zeta_{2}+\frac{5}{2}H_{-1,-1,0}+\frac{5}{2}H_{-1,2}+\frac{9}{2}H_{-1,0}+\frac{39}{2}H_{-2,0}-\frac{473}{12}H_{0}\zeta_{2}-\frac{1853}{49}H_{0,0}$  $\begin{array}{c} -\frac{2}{2}(9_{2})^{3}(9_{2})^{-1}(9_{2})^{-1}(1-1)^{2}(1-1)^{$  $\left[\frac{7}{2}xH_{1} - H_{0,0} + \frac{7}{2}xH_{1,1}\right] + \frac{7}{6}x(1+x)H_{-1,0} + \frac{7}{4}H_{0} - \frac{19}{64}H_{1} + H_{0,0,0} + \frac{5}{6}H_{1,1} + \frac{5}{6}H_{-1,0}$  $\frac{85}{216}$  +  $16C_A^{-2}n_f \left( p_{qg}(x) \left[ 3H_{1,3} + \frac{31}{6}H_{1,0,0} - \frac{17}{2}H_{2,1} + \frac{7}{5}\zeta_2^{-2} - \frac{55}{12}H_{1,1,0} + \frac{31}{12}H_3 - \frac{31}{2}H_1\zeta_3 \right]$  $\frac{5}{12}H_{2,0} - \frac{63}{8}H_{1,0} - \frac{23}{12}H_{1,2} - \frac{155}{6}F_3 + \frac{25}{24}H_2 - \frac{2537}{27}H_0 + \frac{867}{8} - \frac{23}{27}H_{-1,0,0} + 3H_4 - H_{1,1,1}$  $+\frac{383}{77}H_{1,1}-\frac{25}{7}H_{-2,0}-\frac{3}{8}\zeta_2-\frac{7}{4}H_1\zeta_2-3H_{0,0}\zeta_2-\frac{31}{12}H_0\zeta_2+\frac{103}{216}H_1+\frac{5}{2}H_{1,0,0,0}+\frac{2561}{77}H_{0,0}$  $+H_{1} - 2H_{2,0,0} - 3H_{1,0,0} - 5H_{1,0}\zeta_2 + 3H_{0,0,0} - H_{1,1}\zeta_2 - H_{1,1,0,0} - 4H_{1,1,1,0} + 2H_{1,1,1,0}$  $-2H_{1,1,2}-2H_{1,2,0}\Big]+p_{qg}(-x)\Big[H_{-1,-1}\zeta_2-2H_{-1,2}-6H_{-1,-1,0}+H_{1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-1,-1,0}+H_{-1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-1,-1,0}+H_{-1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-1,-1,0}+H_{-1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-1,-1,0}+H_{-1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-1,-1,0}+H_{-1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-1,-1,0}+H_{-1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-1,-1,0}-H_{-1,-1,0}+H_{-1,1,1}+2H_{-2}\zeta_2-H_{-2,0,0}-H_{-2,$  $+\frac{727}{76}H_{-1,0}-H_{-1}\zeta_2-2H_{-2,2}-\frac{5}{2}H_{-1}\zeta_3-H_{-1,-2,0}+2H_{-1,-1,0,0}+2H_{-1,-1,2}-\frac{3}{2}H_{-1,0,0,0}$  $+6H_{-1,-1,-1,0} - 2H_{-1,3} + 2H_{-1,2,1} + (\frac{1}{r} - x^2) \left[\frac{2}{3}H_{2,1} + \frac{32}{9}\zeta_2 - 2H_{1,0,0} + \frac{4}{3}H_{1,1,0} - \frac{10}{9}H_{1,1} + \frac{10}{7}H_{1,1,0} + \frac{10}{9}H_{1,1,0} +$  $-\frac{8}{3}H_{-1,0,0}+\frac{3}{2}H_{1,0}+6\zeta_3+\frac{161}{36}H_1-\frac{2351}{108}\Big]+\frac{2}{3}(\frac{1}{y}+x^2)\Big[\frac{26}{3}H_{-1,0}-\frac{28}{9}H_0-2H_{-1,-1,0}$ 

 $-2H_{-1,2} + H_1\zeta_2 + H_{-1}\zeta_2 + \frac{10}{2}H_2 + H_{1,1,1} + (1-x) \left[15H_{0,0,0,0} - 5H_2\zeta_2 - \frac{65}{2}\zeta_3 + \frac{11}{2}H_{1,1,1} + (1-x)\left[15H_{0,0,0,0} - 5H_2\zeta_3 + \frac{11}{2}H_{1,1,1} + (1-x)\left[15H_{0,0,0,0} - 5H_2\zeta_3 + \frac{11}{2}H_{1,1,1} + (1-x)\left[15H_{0,0,0,0} - 5H_2\zeta_3 + \frac{11}{2}H_{1,1,1} + \frac{11}{2}H_{1,1} + \frac{11}{2}H$  $-\frac{3}{2}H_4 + \frac{5}{2}H_{0,0}\zeta_2 + H_{1,1,0} - \frac{31}{6}H_{2,0} + \frac{17}{12}H_{1,0} - \frac{551}{20}\zeta_2^2 - \frac{29}{4}H_{1,0,0} - \frac{113}{4}H_2 + \frac{18691}{72}H_0$  $+\frac{2243}{108}+\frac{265}{6}H_{-1,0,0}+\frac{33}{2}H_{2,0,0}+19H_{2,1}+\frac{31}{12}H_{1,1}+\frac{23}{2}H_{-2,0}-\frac{497}{26}\zeta_2+\frac{29}{6}H_1\zeta_2-\frac{143}{12}H_3$  $\frac{108}{6} \frac{6}{10} \frac{6}{10} \frac{2}{1223} \frac{12}{10} \frac{12}{10} \frac{2}{10} \frac{12}{10} \frac{2}{10} \frac{12}{10} \frac{2}{10} \frac{10}{10} \frac{10}{10$  $+7H_{-1,-1,0} - \frac{35}{3}H_{1,1,1} - 5H_{-2}\zeta_2 - 11H_{-2,0,0} + \frac{1}{2}H_{-1,0} + \frac{15}{2}H_{-1}\zeta_2 + 8H_{3,1} - 10H_{-2,-1,0}$  $+5H_2\zeta_2 + 4H_{2,1,1} - H_{-3,0} + 36H_0\zeta_3 - 5H_2\zeta_2 + 2H_{-1,2} + 6H_{-1,-1,0} - 6H_{2,1,0} - 3H_{2,1,1}$  $-11H_{0,0,0,0}-5H_{3,1}+\frac{25}{4}H_{1,1,1}+\frac{13}{2}H_{-2}\zeta_{2}+\frac{27}{2}H_{-2,0,0}+\frac{11}{2}H_{-3,0}+\frac{13}{2}H_{2}\zeta_{2}-\frac{17}{4}H_{1,0,0}$ 
$$\begin{split} &-\mathrm{III} \mathbf{k}_{0000} - \mathrm{SH}_{0,1} + \frac{2}{4} \mathbf{k}_{1,1,1} + \frac{2}{3} \mathbf{k}_{1,2} + \frac{2}{3} \frac{1}{4} \mathbf{k}_{1,00} + \frac{2}{3} \mathbf{H}_{1,00} + \frac{2}{3} \mathbf{H}_{1,00} + \frac{2}{3} \mathbf{H}_{1,00} + \frac{2}{3} \mathbf{H}_{1,0} + \frac{2}{3} \mathbf{k}_{1,0} + \frac{2}{3} \mathbf{k}_{1,0$$
 $+\frac{7}{9}xH_2 + \frac{8}{9}xH_{1,0} - \frac{7}{9}x\zeta_2 - (1+x)\left[\frac{3475}{216}H_0 + \frac{103}{12}H_{0,0}\right] + 16C_F^{-2}n_f \left(p_{qg}(x)\left[7H_{1,3} + 7H_4\right]\right)$  $-2H_{-3,0} - 7H_1\zeta_3 + 5H_{2,2} + 6H_{3,0} + 6H_{3,1} + H_{2,1,0} + 4H_{2,0,0} + 3H_{2,1} + 2H_{2,1,1} + \frac{5}{2}H_{2,0}$ 
$$\begin{split} & \underset{A_{1}}{\overset{(1)}{=}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} + \frac{(-\alpha_{2})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} + \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} + \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} + \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}{(-\alpha_{1})_{2}} \frac{(-\alpha_{1})_{2}}}{(-\alpha_{1})_{2}}$$
32 2 2 2 2 2 2 8 5 +11H<sub>0.0</sub> - 2H<sub>1.-2.0</sub> - 7H<sub>1.0</sub> $\zeta_2$  + 3H<sub>1.0.0.0</sub> - 5H<sub>1.1</sub> $\zeta_2$  + 4H<sub>1.1.0.0</sub> + H<sub>1.1.1.0</sub> + 2H<sub>1.1.1.1</sub> + 5H<sub>1.1.2</sub>  $+6\mathbf{H}_{1,2,0}+6\mathbf{H}_{1,2,1}\Big]+4p_{qg}(-x)\Big[\mathbf{H}_{0,0,0,0}-\mathbf{H}_{-2,0}+\mathbf{H}_{-1,-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{\alpha}\mathbf{H}_{-1,0$  $-\frac{5}{4}H_{-1,0,0} - \frac{1}{2}H_{-3,0} + \frac{1}{2}H_{-1}\zeta_2 + H_{-1,-1,0,0} - \frac{1}{4}H_{-1,0,0,0} \Big] + 2(1-x)\Big[H_{2,1,0} - H_{2,0,0} - H_{2,2,0} - H_{2,0,0} - H_{2,$  $-H_{3,1} - 2H_{3,0} - 2H_{-1}\zeta_2 + H_{1,2} - H_{1,0,0} - H_{1,1,0} + H_2\zeta_2 - \zeta_2^2 + \frac{43}{2}H_2 + \frac{49}{2}\zeta_2 + \frac{13}{2}H_{1,1}$  $\begin{array}{c} 33\\ -\frac{33}{16}H_1+\frac{5}{2}H_{1,0}+\frac{7}{2}H_{0,0}\xi_2+\frac{21}{4}\xi_3+\frac{479}{64}-\frac{1}{2}H_{1,1,1}-\frac{1}{2}H_3+\frac{1}{4}H_{2,1}+\frac{1}{2}H_{2,1,1}+\frac{3}{2}H_0\xi_2\\ +\frac{1}{2}H_0\xi_3-\frac{7}{2}H_4+H_1\xi_2-\frac{19}{2}H_{0,0,0}-\frac{239}{16}H_{0,0}-\frac{435}{16}H_0+\frac{435}{2}H_0]+8(1+x)\left[H_{1,-1,0}-H_{-1,0,0}\right]\\ \end{array}$ 

 $P_{gq}^{(2)}(x) = 16C_AC_F n_f \left(\frac{2}{9}x^2 \left[\frac{25}{6}H_1 - \frac{131}{4} + 3\zeta_2 - H_{-1,0} - 3H_2 + H_{1,1} + \frac{125}{6}H_0 - H_{0,0}\right]\right)$  $+\frac{5}{6}p_{99}(x)\left[H_{1,2}+H_{2,1}+\frac{967}{120}+\frac{251}{90}H_1-\frac{39}{10}H_{1,1}-3\zeta_3-\frac{2}{5}H_0\zeta_2-\frac{1}{5}H_1\zeta_2-\frac{4}{3}H_{1,0}+H_{1,1,0}\right]$  $-\frac{2}{5}H_{1,0,0} + H_{1,1,1} + \frac{2}{5}H_{2,0} + \frac{2}{3}p_{00}(-x) \left[2H_{-1}\zeta_2 + \frac{7}{4}\zeta_2 + \frac{41}{12}H_{-1,0} - \frac{151}{72}H_0 + \frac{1}{2}H_{-2,0}\right]$  $+\frac{5}{3}H_2 + 2H_{-1,-1,0} - H_{-1,0,0} - H_{-1,2} + \frac{2}{3}(1-x) \left[H_{-2,0} + 2\zeta_3 - H_3\right] + (1+x) \left[\frac{179}{108}H_1\right]$  $\frac{5}{9}\zeta_{2} + \frac{25}{9}H_{-1,0} - \frac{5}{36}H_{1,1} - \frac{167}{36}H_{0,0} - \frac{1}{3}H_{2,1} - \frac{4}{3}H_{0}\zeta_{2} - \frac{193}{77} + \frac{1}{4}H_{1} + \frac{1}{9}H_{-1,0} + 4H_{2}$  $\frac{7}{-\frac{1}{4}}H_{1,1} + \frac{227}{18}H_0 - \frac{30}{15}H_{0,0} - H_{2,1} - \frac{2}{3}H_0\zeta_2 + \frac{10}{3}H_{-2,0} + 3\zeta_3 + 2H_3 + \frac{2}{3}H_{0,0,0} + x\left[\frac{11}{4}\zeta_2 + \frac{10}{3}H_{-2,0} + \frac{1}{3}\zeta_3 + \frac{10}{3}H_{-2,0} + \frac{10}{3}\zeta_3 + \frac{10}{3}H_{-2,0} + \frac{10}{3}G_{-2,0} + \frac{10}{3}H_{-2,0} + \frac{10}{3}G_{-2,0} + \frac{10}{3}H_{-2,0} + \frac$  $\frac{523}{144} - \frac{19}{36}H_2 + \frac{271}{108}H_0 - \frac{5}{6}H_{1,0} + 16C_A C_F^2 \left(x^2 \left[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1}\right]\right)$  $-6H_2 + 2H_{2,1} + 6\zeta_2 + \frac{335}{54}H_0 - \frac{28}{9}H_{0,0} - \frac{8}{3}H_{0,0,0} + p_{gg}(x) \left[\frac{3}{2}H_1\zeta_3 + \frac{163}{32} - 5\zeta_2 + \frac{27}{4}\zeta_3 + \frac{163}{32}\right]$  $+\frac{6503}{432}H_1 + \frac{2}{9}H_{1,1} + \frac{35}{3}H_{1,1,1} + 4H_2 + \frac{9}{2}H_{2,1} + 4H_{1,0,0} + 2H_{2,0,0} - H_2\zeta_2 + \frac{41}{12}H_{1,2} + H_{2,2}$  $+\frac{191}{24}H_{1,0}+3H_{2,0}-2H_{2,1,1}-\frac{3}{2}H_{-1}\zeta_2-\frac{59}{12}H_1\zeta_2+5H_{1,-2,0}+H_{1,0}\zeta_2+\frac{5}{2}H_{1,0,0,0}-2H_{1,1}\zeta_2$  $+\frac{1}{12}H_{1,1,0}+5H_{1,1,0,0}-3H_{1,1,1,0}-4H_{1,1,1,1}-H_{1,1,2}-2H_{1,2,1}+H_{2,1,0}\Big]+p_{99}(-x)\Big[H_{-1,0}$  $+H_{-1,0}\zeta_2 + \frac{3}{2}H_{-1,0,0} + \frac{27}{10}\zeta_2^2 - 3H_{-1,-1,0} - \frac{11}{2}H_{-1}\zeta_3 - 3H_{-1,-2,0} - \frac{3}{2}H_{-1,0,0,0} - 3H_{-1,2,0}$  $+5H_{-1,-1}\zeta_2 - 4H_{-1,-1,0,0} - 2H_{-1,-1,2} + 6H_{-1,-1,-1,0} + 2H_{-1,2,1}\Big] + (1-x)\Big[H_2\zeta_2 - H_{2,2}$  $+ \frac{23}{12}H_{1,0} - \frac{7061}{23}H_0 - \frac{4631}{14}H_{0,0} - \frac{38}{24}H_{0,0} - H_{-3,0} - 2H_{3,0} - \frac{4433}{23}H_1 - 2H_{2,0,0} - \frac{21}{2}H_{1,0} + \frac{21}{24}H_{1,0} + \frac{21}{24}H_{1,$  $\frac{12}{-\frac{2}{5}}\zeta_{2}^{2} - \frac{7}{2}H_{1,2} + \frac{23}{2}H_{1}\zeta_{2} - 4H_{0}\zeta_{3}\Big] + (1+x)\Big[\frac{49}{6}H_{3} - H_{-2,0} - \frac{432}{56}H_{0}\zeta_{2} - \frac{1}{2}H_{3,1} - \frac{1159}{36}\zeta_{2} - \frac{1}{2}H_{3,1} - \frac$  $+\frac{655}{576} - \frac{151}{6}\zeta_{33} - \frac{185}{18}H_{1,1} + \frac{1}{6}H_{1,1,1} - \frac{95}{9}H_2 + \frac{29}{6}H_{2,1} - \frac{171}{4}H_{-1,0} - 12H_{-1,0,0} + 7H_{-1}\zeta_2$ 576 = 6 = 18 + 16 + 6 + 19 = 76 + 6 + 19 + 6 + 10 + 100 +

 $+\frac{1}{3} p_{gq}(x) \left[ H_{1,2} - H_{1,0} - H_{1} \zeta_{2} + 9 \zeta_{3} + \frac{83}{12} H_{1,1} + 2 H_{-2,0} - \frac{7}{36} H_{1} + 2 H_{0} \zeta_{2} - \frac{1625}{48} + \frac{3}{2} H_{1,0,0} + \frac{3}{2} H_{0,0} + \frac{3}{2$  $+2H_{1,1,0} - \frac{5}{2}H_{1,1,1} + \frac{31}{19}p_{09}(-x)\left[\frac{95}{92}H_0 - \zeta_2 - H_{-1,0}\right] + \frac{1}{2}(2-x)\left[6H_{0,0,0,0} - H_3 - \frac{13051}{299}H_0 - \zeta_2 - H_{-1,0}\right]$  $-\frac{13}{2}\zeta_3 - 4H_{-2,0} - H_{2,0} - \frac{1}{2}H_{1,0} - \frac{1}{2}H_{2,1} + 2H_{0,0,0} - \frac{653}{24}H_{0,0} \Big] + (1+x)\Big[H_0\zeta_2 - \frac{1187}{216}H_0 - \frac{1187}{216}H$  $\frac{1}{8}H_2 - \frac{85}{18}H_{-1,0} - \frac{101}{18}\zeta_2 - \frac{80}{27}H_0 + \frac{23}{18}\zeta_2 - \frac{1}{3}H_{1,1} + \frac{5}{4}xH_{1,1} - \frac{1}{9}H_1 - \frac{37}{17}xH_1 + \frac{210}{18}H_{-1,0}$  $-\frac{23}{9}H_{1,1} - 8H_1\zeta_3 - 6H_{1,-2,0} - 2H_{1,0}\zeta_2 + 3H_{1,1,0} - 3H_{1,1,0,0} - H_{1,1,1,0} + 2H_{1,1,1,1} - 3H_{1,1,2}$  $\frac{8}{-2H_{1,2,0}-2H_{1,2,1}-\frac{9}{2}H_{1,1,1}-\frac{3}{2}H_{1,0,0}-\frac{47}{16}-\frac{47}{16}H_1-\frac{15}{2}\zeta_3]+p_{gg}(-x)\left[2H_{-1,-2,0}-\frac{10}{2}H_{-1,-2,0}-\frac$  $+6H_{-1,-1,0}+3H_{-1}\zeta_{2}+\frac{7}{4}H_{1,0}-\frac{16}{5}\zeta_{2}^{2}-6H_{-1,0,0}-\frac{7}{2}H_{-1,0}+4H_{-1,-1,0,0}-2H_{-1,0}\zeta_{2}$  $4^{-}$  $-H_{-1,0,0,0}$  + (1-x) [9H<sub>1,0,0</sub> + H<sub>1,1,1</sub> - 10H<sub>1</sub> $\zeta_2$  + 3H<sub>0</sub> $\zeta_3$  + H<sub>2,2</sub> - H<sub>2</sub> $\zeta_2$  + H<sub>0,0,0</sub> + 5H<sub>2,0,0</sub>  $-4H_3 + H_{2,1,1} + 3H_{0,0}\zeta_2 + 3H_{3,1} - 3H_4 + \frac{211}{16}H_1 + \frac{49}{20}\zeta_2^2 + (1+x)\left[11\zeta_3 + \frac{1}{4}H_{1,1} + \frac{1}{4}H_{1,0}\right]$  $+\frac{91}{16}H_{0}+36H_{-1,0}+8H_{-1,0,0}-14H_{-1,-1,0}-7H_{-1}\zeta_{2}+2H_{1,2}+4H_{0}\zeta_{2}-H_{2,1}+2H_{-2,0,0}$  $\frac{10}{+5H_{-2,0} + \frac{11}{2}H_2 - 2H_{0,0,0,0}} - 2H_{-1,-1,0} - H_{-1}\zeta_2 - \frac{13}{4}\zeta_2 + \frac{9}{4}H_{1,0} + \frac{9}{20}\zeta_2^2 + \frac{287}{23} + \frac{11}{16}H_1$  $+4H_{-1,0,0} + 16H_{-3,0} - 4H_{-2}\zeta_2 - 8H_{-2,-1,0} - 5H_2\zeta_2 + \frac{19}{4}H_2 + H_{2,2} - \frac{35}{8}H_{0,0} + 9H_0\zeta_3$  $+25H_{-2,0}+6H_{-2,0,0}+\frac{3}{2}x\Big[\frac{58}{2}\zeta_{2}-\frac{7}{2}H_{1}\zeta_{2}+4H_{1,1}-\frac{3}{2}H_{1,1,1}+\frac{5}{2}H_{1,0,0}-\frac{175}{06}+H_{3,1}+\frac{19}{2}\zeta_{2}+\frac{1}{2}H_{1,1,1$  $+2H_{2,0}-14H_0+H_{0,0}\zeta_2-H_{-1,0}-H_4-\frac{3}{2}H_{2,1}+\frac{1}{2}H_{2,1,1}+3H_{2,0,0}-\frac{5}{z}H_3-H_{1,2}-\frac{7}{z}H_0\zeta_2$  $+\frac{2}{2}H_{1,1,0}-\frac{29}{2}H_{0,0,0}-\frac{185}{2}H_{0,0}]$ 

$$\begin{split} p_{12}^{(2)}(z) &= 16\mathcal{L}_{0}\mathcal{L}_{0}^{(2)}\left(\hat{z}^{2}\Big[\frac{2}{3}u_{2}+3u_{1,0}-\frac{97}{12}u_{1}+\frac{8}{3}u_{-2,0}-\frac{2}{3}u_{5,2}-\frac{13}{20}u_{5,0}-\frac{16}{3}v_{2,2}+2u_{1}\right)\\ &-6u_{-1,0,2}+2u_{2,1}+\frac{127}{14}u_{0,0}-\frac{11}{12}\Big]+p_{00}(z)\Big[2\zeta_{2}-\frac{8}{22}\Big]+\frac{4}{3}(\frac{1}{z}-z^{2}\Big)\Big[\frac{12}{2}u_{1,0}-\frac{43}{3}u_{1,0}\\ &-\frac{12}{14}u_{1,0}-\frac{6}{3}u_{1,2}-\frac{3}{2}u_{1,1}+2u_{1,2}+4u_{2,2}-2u_{1,0}+\frac{112}{12}u_{1,1}-4u_{1,1,0}-1u_{1,1,1}-\frac{173}{12}u_{1,1}\\ &+6u_{-1,0}+8u_{5,1}-6u_{-2,0}-\frac{5}{3}u_{1,2}-\frac{9}{2}u_{1,2}-\frac{43}{2}u_{1,1}+\frac{43}{12}z+\frac{1}{12}u_{1,2}-3u_{1,1,0}-4u_{1,0,0}\\ &-\frac{12}{12}u_{0,0}+\frac{2}{2}\zeta_{1}-u_{1,1}+\frac{12}{12}u_{1,1}-2u_{1,1}-\frac{14}{12}u_{1,1}-4u_{1,1,0}-4u_{1,1,1,0}-4u_{1,1,1,0}-4u_{1,1,1,0}-4u_{1,1,1,1,1}-4u_{1,1,1,1}-4u_{1,1,1,1}-4u_{1,1,1,1}$$

 $+12H_{0,0,0,0}-\frac{293}{109}+\frac{61}{4}H_0\zeta_2-\frac{7}{2}H_{1,0}-\frac{857}{26}H_1-9H_0\zeta_3+16H_{-2,-1,0}-4H_{-2,0,0}+8H_{-2}\zeta_2$  $-\frac{13}{2}H_{1,0,0} + \frac{3}{4}H_{1,1} - H_{1,1,0} - H_{1,1,1} + (1+x)\left[\frac{1}{6}H_{2,0} - \frac{95}{2}H_{-1,0} - \frac{149}{26}H_2 + \frac{3451}{100}H_0\right]$  $-7H_{-2,0} + \frac{302}{10}H_{0,0} + \frac{19}{6}H_3 - \frac{991}{26}\zeta_2 - \frac{163}{6}\zeta_3 - \frac{35}{27}H_{0,0,0} + \frac{17}{6}H_{2,1} - \frac{43}{10}\zeta_2^2 + 13H_{-1}\zeta_2$  $\begin{array}{l} -m_{-20} + \frac{1}{9} \frac{1}{100} \frac{1}{9} - \frac{1}{100} \frac{1}{9} - \frac{1}{30} \frac{1}{9} - \frac{1}{9} \frac{1}{9} \frac{1}{9} - \frac{1}{100} \frac{1}{9} + \frac{1}{100} \frac{1}{100} + \frac{1}{100} \frac{1}{100} \frac{1}{100} + \frac{1}{100} \frac$  $+16C_{A}^{2}n_{f}\left(x^{2}\left[\zeta_{3}+\frac{11}{9}\zeta_{2}+\frac{11}{9}H_{0,0}-\frac{2}{3}H_{3}+\frac{2}{3}H_{0}\zeta_{2}+\frac{1639}{108}H_{0}-2H_{-2,0}\right]+\frac{1}{3}\rho_{BS}(x)\left[\frac{10}{3}\zeta_{2}+\frac{1}{9}H_{0,0}-\frac{2}{3}H_{0,0}+\frac{2}{3}H_{0,0}$  $-\frac{209}{36} - 8\zeta_3 - 2H_{-2,0} - \frac{1}{2}H_0 - \frac{10}{3}H_{0,0} - \frac{20}{3}H_{1,0} - H_{1,0,0} - \frac{20}{3}H_2 - H_3 \Big] + \frac{10}{9}p_{gg}(-x)\Big[\zeta_2 - \frac{1}{3}H_2 - H_3\Big] + \frac{10}{9}p_{gg}(-x)\Big[\zeta_2 - \frac{1}{3}H_3 - H_3\Big] + \frac{10}{9}p_{gg}(-x)\Big[\zeta_2 - \frac{1}{9}H_3 - H_3\Big] +$ 
$$\begin{split} & -\frac{1}{36} - \frac{8}{5}_3 - \frac{241}{20} - \frac{2}{3} \frac{140}{3} - \frac{2}{3} \frac{140}{10} - \frac{2}{3} \frac{141}{10} - \frac{114}{100} - \frac{2}{3} \frac{142}{10} - \frac{141}{3} + \frac{2}{9} \frac{142}{100} - \frac{141}{3} \frac{141}{100} + \frac{141}{30} + \frac{141}{300} - \frac{141}{300} + \frac{141}{300} - \frac{141}{300}$$
 $\begin{array}{c} \overline{3}^{-1}\overline{41}_{(1)} (-11,10) - 1 & \chi^{-1} - 1 & \underline{54}^{-1} & \chi^{-1} - \frac{7}{3} & \underline{51}^{-1} & \underline{51}^{ +\frac{39}{20}\zeta_{2}^{2}-\frac{7}{12}H_{3}-\frac{53}{9}H_{0,0}+\frac{7}{12}H_{0}\zeta_{2}-\frac{5}{2}H_{0,0}\zeta_{2}+5\zeta_{3}-7H_{-1,-1,0}+\frac{77}{6}H_{-1,0}+\frac{9}{2}H_{-1,0,0}$  $+2H_{-1,2} - 3H_2\zeta_2 - \frac{2}{3}H_{2,0} + \frac{3}{2}H_{2,0,0} + \frac{3}{2}H_4 + \frac{1}{9}\zeta_2 + 7H_{-2,0} + 2H_2 + \frac{458}{27}H_0 + H_{0,0}\zeta_2$  $+\frac{3}{2}\zeta_{2}^{2}+4H_{-3,0}-x[\frac{131}{12}H_{0,0}-\frac{8}{3}H_{0}\zeta_{2}+\frac{7}{2}H_{3}-H_{0,0,0,0}+\frac{7}{6}H_{0,0,0}+\frac{194}{216}H_{0}+6H_{0}\zeta_{3}]$  $\begin{array}{c} \frac{1}{2} e^{-1} \left( \frac{123}{28} + \frac{1}{5} e^{-1} + \frac{1}{2} \right) e^{-1} e^{-1}$  $-8H_{0,0}\zeta_2 + 4H_{0,0,0,0} - 6H_1\zeta_3 - 4H_{1,-2,0} + 10H_{2,0,0} - 6H_{1,0}\zeta_2 + 8H_{1,0,0,0} + 8H_{1,1,0,0} + 8H_4$  $\frac{134}{n}H_{1,0} + \frac{11}{4}H_{1,0,0} + 8H_{1,2,0} + 8H_{1,3,1} + \frac{134}{9}H_2 - 4H_2\zeta_2 + 8H_{3,1} + 8H_{2,2} + \frac{11}{6}H_3 + 10H_{3,0}$  $+8H_{2,1,0}] + p_{gg}(-x) \left[\frac{11}{2}\zeta_{2}^{2} - \frac{11}{2}H_{0}\zeta_{2} - 4H_{-3,0} + 16H_{-2}\zeta_{2} - 12H_{-2,2} - \frac{134}{2}H_{-1,0} + 2H_{2}\zeta_{2}\right]$  $\begin{array}{c} + 6H_{2,1,0} \right)^{-1} P_{RS}(-x) \left[ \frac{1}{2} \frac{52}{52} - \frac{1}{6} - \frac{1}{9} \frac{52}{24} - \frac{1}{6} - \frac{1}{9} \frac{5}{24} - \frac{1}{2} \frac{5}{24} - \frac{1}{2} \frac{5}{24} + \frac{1}{2} \frac{5}{24} - \frac{1}{2} \frac{5}{24} + \frac{1}{2} \frac{1}{2} \frac{5}{24} + \frac{1}{2} \frac{1}{2} \frac{1}{24} + \frac{1}{2} \frac$  $+18H_{-1,0}\zeta_2-16H_{-1,0,0,0}-4H_{-1,2,0}-16H_{-1,3}-5H_0\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0,0}\zeta_3-8H_{0,0}\zeta_2+4H_{0,0,0,0}+2H_{3,0}\zeta_3-8H_{0$  $-\frac{67}{9}\zeta_{2}+\frac{67}{9}H_{0,0}+8H_{4}]+\left(\frac{1}{\tau}-x^{2}\right)\left[\frac{16619}{167}+\frac{22}{3}H_{2,0}-\frac{55}{2}\zeta_{3}-\frac{11}{2}H_{0}\zeta_{2}-\frac{67}{9}H_{2}-\frac{67}{9}H_{1,0}-\frac{67}{9}H_{1,0}-\frac{67}{9}H_{2}-\frac$ 

 $-\frac{413}{109}H_1 - \frac{11}{2}H_1\zeta_2 + \frac{33}{2}H_{1,0,0} + 11(\frac{1}{2} + x^2)\left[\frac{71}{64}H_0 - \frac{1}{6}H_3 - \frac{389}{109}\zeta_2 - \frac{2}{2}H_{-2,0} - \frac{1}{2}H_{-1}\zeta_2\right]$  $+H_{-1,-1,0} - \frac{523}{108}H_{-1,0} + \frac{8}{2}H_{-1,0,0} + H_{-1,2} + (1-x) \left[\frac{31}{22}H_1 + \frac{27}{27}H_{1,0} - \frac{25}{2}H_{1,0,0} - 4H_{-3,0}\right]$  $-\frac{263}{12}H_{0,0} - \frac{29}{3}H_{0,0,0} - \frac{19}{3}H_{-2,0} - \frac{11317}{108} - 4H_{-2}\zeta_2 - 8H_{-2,-1,0} - 12H_{-2,0,0} - \frac{3}{2}H_1\zeta_2$  $\begin{array}{c} 12 \\ +(1+x) \Big[ \frac{27}{2} H_0 \zeta_2 - \frac{43}{6} H_3 + \frac{2}{3} H_{2,0} + \frac{4651}{216} H_0 - \frac{329}{18} \zeta_2 + \frac{11}{2} (1+x) \zeta_3 - \frac{43}{5} \zeta_2^2 - \frac{215}{6} H_{-1,0} \\ -22 H_{0,0} \zeta_2 - 8 H_0 \zeta_3 - 3 H_{-1,-1,0} + 38 H_{-1,0,0} + 25 H_{-1,2} + 10 H_{2,0,0} - 4 H_2 \zeta_2 + 16 H_{3,0} + 26 H_4 \\ \end{array}$  $-\frac{158}{9}H_2 - \frac{53}{2}H_{-1}\zeta_2 - 29H_{0,0} - \frac{40}{2}H_{0,0,0} + 27H_{-2,0} + \frac{41}{2}H_0\zeta_2 - 20H_3 - 24H_{2,0} + \frac{53}{6}\zeta_2$  $+\frac{601}{12}H_0 + 24\zeta_3 + 2\zeta_2^2 + 27H_2 - 4H_{0,0}\zeta_2 - 16H_0\zeta_3 - 16H_{-3,0} + 28xH_{0,0,0,0} + \delta(1-x)\left[\frac{79}{32}\right]$  $\frac{12}{-\zeta_2\zeta_3} + \frac{1}{\kappa}\zeta_2 + \frac{11}{74}\zeta_2^2 + \frac{67}{6}\zeta_3 - 5\zeta_5 \Big] + 16C_Fn/^2 \Big(\frac{2}{9}x^2 \Big[\frac{11}{6}H_0 + H_2 - \zeta_2 + 2H_{0,0} - 9\Big] + \frac{1}{3}H_2 - \frac{1}{3}$ 
$$\begin{split} &\frac{1}{3} \sum_{i=1}^{i} - \frac{10}{3} H_0 - \frac{1}{3} H_{0,0} + 2 + \frac{2}{9} \left(\frac{1}{s} - x^2\right) \left[\frac{8}{3} H_1 - 2H_{1,0} - H_{1,1} - \frac{77}{18}\right] - (1-x) \left[\frac{1}{3} H_{1,0} + \frac{1}{6} H_{1,1} + \frac{1}{9} H_{1,0} + \frac{1}{3} H_{1,0} + \frac{1}{6} H_{1,1} + \frac{1}{9} H_{1,1} + \frac{1}{9}$$
 $-H_{2,1}-2H_{2,0}]+\frac{11}{144}\delta(1-x)\Big)+16C_F^2n_f\Big(\frac{4}{3}x^2\Big[\frac{163}{16}+\frac{27}{8}H_0+\frac{7}{2}H_{0,0}-H_{2,0}-\zeta_2+\frac{9}{4}H_{1,0}\Big]$  $-H_{2,1} + \frac{1}{2}H_{0,0,0} + \frac{45}{16}H_1 + H_2 - 2H_{-2,0} - \frac{3}{2}\zeta_3 \Big] + \frac{4}{3}(\frac{1}{x} - x^2) \Big[\frac{31}{16}H_1 - \frac{11}{16} - \frac{5}{4}H_{1,0} + \frac{1}{2}H_{1,0,0} + \frac{1}{2}H_{1,0,0$  $-H_1\zeta_2 - H_{1,1} + H_{1,1,0} + H_{1,1,1} + \zeta_3 + \frac{4}{3}(\frac{1}{x} + x^2) [H_{-1}\zeta_2 + 2H_{-1,-1,0} - H_{-1,0,0}] + \frac{215}{12}H_{0,0}$  $+\frac{20}{^{2}}H_{0}-\frac{131}{^{6}}+3H_{2,0}+\frac{205}{^{17}}x\zeta_{2}-3H_{1,0}+H_{2,1}-\frac{85}{^{17}}H_{1}+\frac{11}{^{4}}H_{2}+8H_{-2,0}+2\zeta_{2}{^{2}}-H_{0}\zeta_{2}$  $+H_3 + 6H_0\zeta_3 + 8H_{-3,0} - 4xH_{0,0,0} + (1-x)\left[\frac{107}{12}H_1 - \frac{5}{6}H_{1,0} - 4\zeta_2 + H_0\zeta_3 - 8H_{-2,-1,0}\right]$  $-4H_{-2}\zeta_{2} + 4H_{-2,0,0} - 4H_{1}\zeta_{2} + \frac{7}{2}H_{1,0,0} - \frac{7}{12}H_{1,1} + H_{1,1,0} + H_{1,1,1} + (1+x)\left[\frac{5}{4}H_{2} + \frac{33}{8}\right]$  $-\frac{99}{4}H_{0,0}-8H_{2,0}-\frac{541}{24}H_0-4H_{2,1}-\frac{3}{2}H_{0,0,0}-2x\zeta_3+\frac{9}{2}\zeta_2{}^2+5H_0\zeta_2-5H_3-4H_{-1}\zeta_2$  $-8H_{-1,-1,0} + \frac{67}{2}H_{-1,0} + 4H_{-1,0,0} + 2H_{0,0}\zeta_2 - 2H_{0,0,0,0} - 4H_2\zeta_2 + 3H_{2,0,0} + 2H_{2,1,0}$  $+2H_{2,1,1}+H_{3,1}-2H_4 + \frac{1}{16}\delta(1-x)$ 



 Left: P<sup>(2)</sup><sub>gq</sub> comparison of exact result and estimates from fixed moments Larin, Nogueira, van Ritbergen, Vermaseren '97; Retey, Vermaseren '00; van Neerven, Vogt '00
 Right: same for P<sup>(2)</sup><sub>gg</sub>

### **Easy-to-use parametrization**

- Combine exact limits for  $x \rightarrow 0$  and  $x \rightarrow 1$  with smooth fit for intermediate  $x \rightarrow 1$  fit quality better than one per mille
- Notation : end-point logarithms  $L_0 = \ln(x)$ ,  $L_1 = \ln(1-x)$ , +-distribution  $\mathcal{D}_0 = 1/(1-x)_+$

$$\begin{split} P_{\rm gg}^{(2)}(x) &\cong +2643.521\,\mathcal{D}_0 + 4425.894\,\delta(1-x) + 3589\,L_1 - 20852 + 3968\,x - 3363\,x^2 \\ &+ 4848\,x^3 + L_0L_1\,(7305 + 8757\,L_0) + 274.4\,L_0 - 7471\,L_0^2 + 72\,L_0^3 - 144\,L_0^4 \\ &+ 14214\,x^{-1} + 2675.8\,x^{-1}L_0 \\ &+ n_f\left(-412.172\,\mathcal{D}_0 - 528.723\,\delta(1-x) - 320\,L_1 - 350.2 + 755.7\,x - 713.8\,x^2 \\ &+ 559.3\,x^3 + L_0L_1\,(26.15 - 808.7\,L_0) + 1541\,L_0 + 491.3\,L_0^2 + 832/9\,L_0^3 \\ &+ 512/27\,L_0^4 + 182.96\,x^{-1} + 157.27\,x^{-1}L_0\right) \\ &+ n_f^2\left(-16/9\,\mathcal{D}_0 + 6.4630\,\delta(1-x) - 13.878 + 153.4\,x - 187.7\,x^2 + 52.75\,x^3 \\ &- L_0L_1\,(115.6 - 85.25\,x + 63.23\,L_0) - 3.422\,L_0 + 9.680\,L_0^2 - 32/27\,L_0^3 \\ &- 680/243\,x^{-1}\right). \end{split}$$

### The large *x*-limit : $x \rightarrow 1$

- Large *x*-limit for diagonal splitting functions  $P_{aa}^{(2)}$ , a = q, g

$$P_{\mathrm{aa},\to 1}^{(2)}(x) = \frac{A_3^{\mathrm{a}}}{(1-x)_+} + B_3^{\mathrm{a}}\delta(1-x) + C_3^{\mathrm{a}}\ln(1-x) + O(1)$$

- Result for  $A_3^a$  is new  $\longrightarrow$  important for threshold resummation in soft/collinear limit

one-loop  $A_1^{\mathrm{q}} = 4C_F$ 

two-loop 
$$A_2^{q} = 8C_F C_A \left(\frac{67}{18} - \zeta_2\right) - \frac{5}{9}C_F n_f$$

three-loop 
$$A_3^{q} = 16C_F C_A^2 \left(\frac{245}{24} - \frac{67}{9}\zeta_2 + \frac{11}{6}\zeta_3 + \frac{11}{5}\zeta_2^2\right) + 16C_F^2 n_f \left(-\frac{55}{24} + 2\zeta_3\right) + 16C_F C_A n_f \left(-\frac{209}{108} + \frac{10}{9}\zeta_2 - \frac{7}{3}\zeta_3\right) + 16C_F n_f^2 \left(-\frac{1}{27}\right)$$

- Verify expected relation  $A_3^{\rm g} = \frac{C_A}{C_E} A_3^{\rm g}$
- Surprising relation for subleading logarithms

$$C_1^{\rm a} = 0 , \quad C_2^{\rm a} = (A_1^{\rm a})^2 , \quad C_3^{\rm a} = 2A_1^{\rm a}A_2^{\rm a}$$



– For  $N 
ightarrow \infty$ , diagonal anomalous dimensions grow as  $\ln N$ 

- Left : perturbative expansion of  $\gamma_{qq}(N)$  (and  $\gamma_{ps}(N)$  seperately) with  $n_f = 4$  and  $\alpha_s(\mu^2) = 0.2$
- Right : same for  $\gamma_{qq}(N)$



- Left : perturbative expansion of  $\gamma_{
  m qg}(N)$  with  $n_f = 4$  and  $\alpha_s(\mu^2) = 0.2$
- Right : same for  $\gamma_{
  m gq}(N)$

## The small *x*-limit : $x \rightarrow 0$

- General structure at small x

$$P_{\mathrm{aa},\to0}^{(2)}(x) = = E_1^{\mathrm{ab}} \frac{\ln x}{x} + E_2^{\mathrm{ab}} \frac{1}{x} + O(\ln^4 x)$$

$$E_{1}^{qq} = -\frac{896}{27}C_{A}C_{F}n_{f}$$

$$E_{2}^{qq} = \left[-\frac{27044}{81} + \frac{512}{9}\zeta_{2} + 96\zeta_{3}\right]C_{A}C_{F}n_{f} + \left[\frac{220}{3} - 64\zeta_{3}\right]C_{F}^{2}n_{f} + \frac{64}{27}C_{F}n_{f}^{2}$$

$$E_{1}^{\text{qg}} = -\frac{896}{27}C_{A}^{2}n_{f} = \frac{C_{A}}{C_{F}}E_{1}^{\text{qq}}$$

$$E_{2}^{\text{qg}} = \left[-\frac{9404}{27} + \frac{512}{9}\zeta_{2} + 96\zeta_{3}\right]C_{A}^{2}n_{f} + \left[\frac{220}{3} - 64\zeta_{3}\right]C_{A}C_{F}n_{f} - \frac{424}{81}C_{A}n_{f}^{2} + \frac{1232}{81}C_{F}n_{f}^{2}$$

- Coefficients  $E_1^{\rm qq}$  and  $E_1^{\rm qg}$  agree with prediction from the small-*x* resummation Catani, Hautmann '94



- Exact result, estimates from fixed moments and leading small-*x* term - Pure-singlet splitting function  $P_{\rm ps}^{(2)}$  (left) and  $P_{\rm qg}^{(2)}$  (right)

$$E_{1}^{gg} = \left[\frac{6320}{27} - \frac{176}{3}\zeta_{2} - 32\zeta_{3}\right]C_{A}^{3} + \left[\frac{1136}{27} - \frac{32}{3}\zeta_{2}\right]C_{A}^{2}n_{f} - \left[\frac{1376}{27} - \frac{64}{3}\zeta_{2}\right]C_{A}C_{F}n_{f}$$

$$E_{2}^{gg} = \left[\frac{146182}{81} - \frac{3112}{9}\zeta_{2} - \frac{1144}{3}\zeta_{3} - \frac{464}{5}\zeta_{2}^{2}\right]C_{A}^{3} + \left[\frac{19264}{81} - \frac{128}{3}\zeta_{2} - \frac{176}{3}\zeta_{3}\right]C_{A}^{2}n_{f}$$

$$- \left[\frac{30662}{81} - \frac{608}{9}\zeta_{2} - \frac{224}{3}\zeta_{3}\right]C_{A}C_{F}n_{f} - \left[\frac{44}{3} - \frac{64}{3}\zeta_{3}\right]C_{F}^{2}n_{f} + \frac{472}{81}C_{A}n_{f}^{2} - \frac{1232}{81}C_{F}n_{f}^{2}$$

- No logarithm  $\ln^2 x/x$  in  $P_{gg}^{(2)}$  as predicted by leading logarithmic BFKL equation Kuraev, Lipatov, Fadin '77; Balitsky, L.N. Lipatov '78; Jaroszewicz '82
- Coefficient  $E_1^{\text{gg}}$  agrees with prediction from next-to-leading logarithmic BFKL equation Fadin, Lipatov '98
- New coefficients  $E_1^{
  m gq}$ ,  $E_2^{
  m gq}$  with interesting relation

$$E_{1}^{\text{gg}} = \frac{C_{A}}{C_{F}} E_{1}^{\text{gq}} - \frac{8}{3} C_{A} n_{f} (C_{A} - 2C_{F}) \rightarrow \frac{C_{A}}{C_{F}} E_{1}^{\text{gq}} - \frac{8}{3} n_{f} \text{ for SU(N)}$$



Exact result, estimates from fixed moments and leading small-x term

- Splitting function  $P_{\rm gq}^{(2)}$  (left) and  $P_{\rm gg}^{(2)}$  (right)



Convolution of P<sup>(2)</sup><sub>gg</sub> with schematic 'steep' (left) and 'flat' gluon distributions
 Comparison of exact result for P<sup>(2)</sup><sub>gg</sub> with various approximations of small-*x* terms

### Numerical implications for singlet distributions

- Perturbative expansion of logarithmic scale derivative

$$\frac{d}{d\ln\mu_f^2}f(x,\mu_f^2) = \left[P(\alpha_s(\mu_f^2)) \otimes f(\mu_f^2)\right](x)$$

- Parametrization of singlet quark distribution

$$xq_{s}(x,\mu_{0}^{2}) = 0.6x^{-0.3}(1-x)^{3.5}(1+5.0x^{0.8})$$
  
$$xg(x,\mu_{0}^{2}) = 1.6x^{-0.3}(1-x)^{4.5}(1-0.6x^{0.3})$$

- Default values 
$$n_f = 4$$
 and  $\alpha_s(\mu^2) = 0.2$ 



- Perturbative expansion of logarithmic scale derivative  $d \ln q_s/d \ln \mu_f^2$ 

- Scale derivative dominated at large x by  $P_{
  m qq}\otimes q_{
  m s}$ , at small x by  $P_{
  m qg}\otimes g$  contribution
- $P_{
  m qq} \otimes q_{
  m s}$  contribution negligible at small x,  $P_{
  m qg} \otimes g$  contribution negligible at large x



- Perturbative expansion of logarithmic scale derivative  $d \ln g / d \ln \mu_f^2$ 

- Scale derivative dominated by  $P_{
m gg}\otimes g$  at all x, however  $P_{
m gq}\otimes q_{
m s}$  nowhere negligible



- Singlet-quark distribution : NLO and NNLO dependence of  $\dot{q}_s \equiv d \ln q_s / d \ln \mu_f^2$  on renormalization scale  $\mu_r$ 



- Gluon distribution :

NLO and NNLO dependence of  $\dot{q}_{
m s}\equiv d\ln q_{
m s}/d\ln \mu_f^2$  on renormalization scale  $\mu_r$ 



$$\Delta \dot{f} \equiv \frac{\max\left[\dot{f}(x,\mu_r = \frac{1}{2}\mu_f \dots 2\mu_f)\right] - \min\left[\dot{f}(x,\mu_r = \frac{1}{2}\mu_f \dots 2\mu_f)\right]}{2\left|\operatorname{average}\left[\dot{f}(x,\mu_r = \frac{1}{2}\mu_f \dots 2\mu_f)\right]\right|}$$

- NLO and NNLO prediction for  $\Delta \dot{q}_{\rm s}$  (left) and for  $\Delta \dot{g}$  (right)

## **The Summary**

## **Motivation**

- NNLO analysis of deep-inelastic structure functions  $F_2, F_3 \longrightarrow$  high precision
  - stability under scale variations at NNLO
  - match experimental accuracy in final HERA data
  - NNLO parton distributions for LHC precision analyses

### **Methods**

- Mellin moments and nested sums —> powerful technology
  - apply innovative and efficient method to solve multi-loop integrals
  - formalism with wide range of applications

### Upshot

- Phenomenology for deep-inelastic scattering and hard hadronic interactions
  - reach new level of precision