Lattice QCD at finite chemical potential

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A comparison with the resonance gas model

QCD-phase diagram: From theory to experiment

Skopelos, Hallas, 28 May - 3 June, 2004

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Outline

- Introduction
- Lattice QCD at finite T and μ
- Ideal resonance gas model at $\mu \neq 0$
- Summary

Hadron - QGP

- Finite intrinsic spatial extension of hadrons: **Pomeranchuk (1913-1966)** 1951

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- **Lattice regularized QCD** at $\mu = 0$ since 1975 and at $\mu \neq 0$ since 1999

QCD phase diagram

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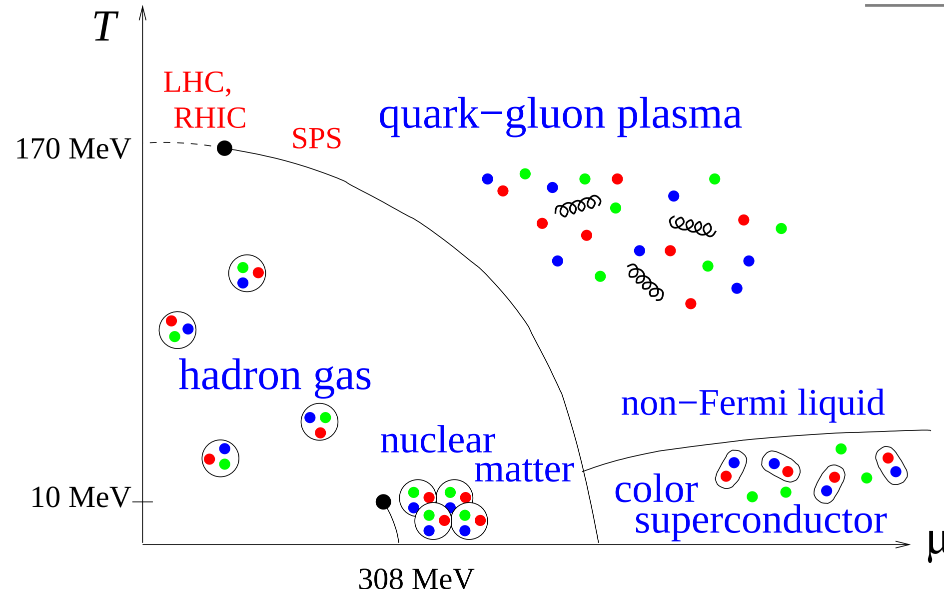
Finite μ QCD: attracting considerable attention in High energy physics, nuclear physics and astrophysics

Theorists: $T - \mu$ diagram has a rich structure

Experimentalists: reveal it through different methods

* Effective Models like Nambu-Jona-Lasino (NJL)
* Lattice regularized QCD: non-perturbative QCD
* Statistical models, like Bootstrap



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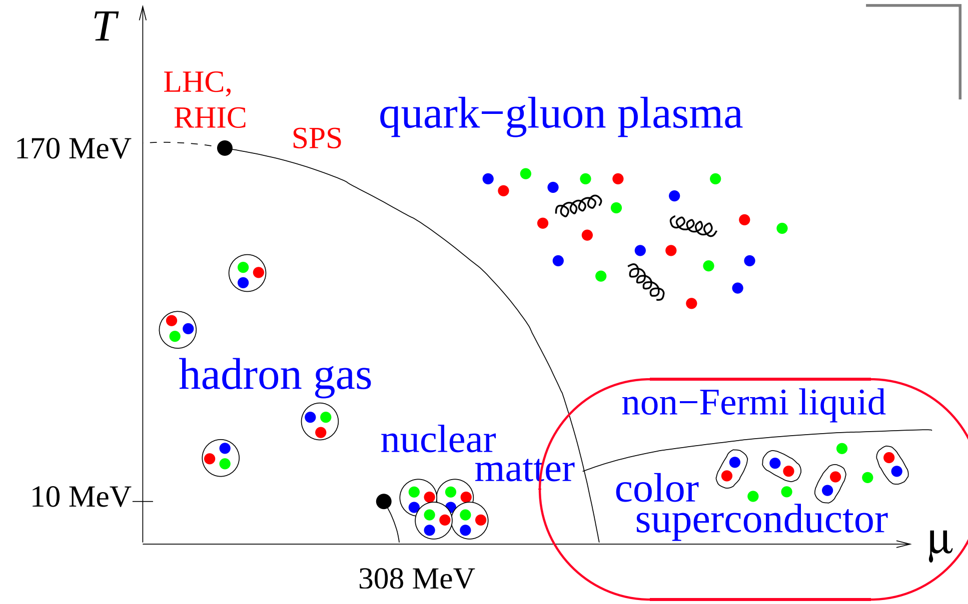

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At large μ confi ned hadron matter is conjectured to move to phases of Color Superconductivity:

one-gluon exchange calculations:QCD CSC [Barrois (NPB129:390), Bailin and Love (PR107:325)]
attractive forces near Fermi surface: Cooper pair [Iwasaki, Iwado (PLB350:163)]
instanton models: 100 MeV gap energy, [Alford, Rajagopal, Wilczek, PLB422:247]
2SC for 2 flavors [Rapp, Schäfer, Shuryak, Velkovsky PL81:53]
CFL for 3 flavors. Pairing rasults in $p \neq 0$ [Alford, Bowers, Rajagopal, PRD63:074016]

[see M. Buballa]

Lattice regularized QCD

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Non-perturbative implementation of field theory using Feymann path integral approach:

$$\mathcal{Z} \quad = \quad \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \; \exp(-S) = \int \mathcal{D}A_\mu \det \mathcal{M} \exp\left(\int d^4 \, x \, \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right)$$

Physical observables: [McLerran, Gavai]

$$< \mathcal{O} > \quad = \quad \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \mathcal{O} \; \exp(-S)$$

Kenneth Wilson: Euclidian gauge theories on the lattice to study confinement and non-perturbative QCD 1974

- Space-time discretization, link variable $U_{\mu,\nu}$
- Lattice transcription of field variables, $\psi(n)$, $A_\mu(n)$
- Construction of the action S
- Definition of the measure of integration in $\mathcal{D}$
- Transcription of operators $\mathcal{O}$ into physical units

$U_\mu(n) = \exp(iag \int_{na}^{(n+\hat{n})a} dz \; A_\mu(z))$, $\qquad U_\mu^\dagger(n) = U_{-\mu}(n+\hat{\mu})$,

$U_{\mu,\nu}(n) = U_\mu(n) U_\nu(n + a\hat{\mu}) U_\mu^\dagger(n + a\hat{\mu}) U_\mu^\dagger(n)$

Lattice QCD at $\mu \neq 0$

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With a naive discretization of the fermionic action for the free theory with $\mu \neq 0$ we get

$$\mathcal{Z} = \int \prod_x d\psi_x \, d\bar{\psi}_x \; \exp(-S_F)$$

$$S_F = a^3 \sum_x \left(ma\bar{\psi}_x\psi_x + \mu a\bar{\psi}_x\gamma_4\psi_x + \frac{1}{2}\sum_{j=1}^{4}(\bar{\psi}_x\gamma_j\psi_{x+\hat{j}} - \bar{\psi}_{x+\hat{j}}\gamma_j\psi_x) \right)$$

In continuum limit it leads to a quadratic divergence in $\epsilon = -\partial \ln \mathcal{Z}/\partial(1/T) \simeq (\mu/a)^2$!

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Successful prescription for including μ has been given in Hasenfratz, Karsch, PLB125:308, where the last two terms should be replaced by

$$\frac{1}{2}\sum_{j=1}^{3}(\bar{\psi}_x\gamma_j\psi_{x+\hat{j}} - \bar{\psi}_{x+\hat{j}}\gamma_j\psi_x) + \frac{1}{2}(e^{\mu a}\bar{\psi}_x\gamma_4\psi_{x+\hat{4}} - e^{-\mu a}\bar{\psi}_{x+\hat{4}}\gamma_4\psi_x)$$

Wilson action $S_F(\mu a) = \sum_x (\bar{\psi}_x\psi_x - \kappa \sum_{j=1}^{3}[\bar{\psi}_x(1-\gamma_j)U_{x,j}\psi_{x,\hat{j}} + \bar{\psi}_{x+\hat{j}}(1+\gamma_j)U^{\dagger}_{x,j}\psi_x]$

$$-\kappa[e^{\mu a}\bar{\psi}_x(1-\gamma_4)U_{x,4}\psi_{x,\hat{4}} + e^{-\mu a}\bar{\psi}_{x+\hat{4}}(1+\gamma_4)U^{\dagger}_{x,4}\psi_x])$$

The first lattice simulation at $\mu \neq 0$ is color $SU(2)$ in 1984 [A. Nakamura, PLB149:391]
$T_c \simeq 200 - 250$ MeV with a relative large error.

Monte Carlo simulation at $\mu \neq 0$

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Including dynamical fermions in LQCD is usually achieved through integrating them out. This leads to determinant of the fermion matrix (Grassmann properties) and effective action depending on the gauge fields

$$\mathcal{Z} = \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_G(U)-\bar{\psi}\mathcal{M}(U)\psi} = \int \mathcal{D}U \det \mathcal{M}(U)\, e^{-S_G(U)}$$

Because of the probabilistic interpretation of the path intengral, Monte Carlo simulations are possible for positive integrand: $\mathcal{M}$ similar to its adjoint $\mathcal{M}^\dagger = A\mathcal{M}A^{-1}$

$$\mathcal{M}_{x,y} = \delta_{x,y} - \kappa \sum_{j=1}^{3} [(r-\gamma_j)U_{x,j}\delta_{x,y-\hat{j}} + (r+\gamma_j)U^\dagger_{x,j}\delta_{x,y+\hat{j}}]$$
$$-\kappa[e^{\mu a}(r-\gamma_4)U_{x,4}\delta_{x,y-\hat{4}} + e^{-\mu a}(r+\gamma_4)U^\dagger_{x,4}\delta_{x,y+\hat{4}}]$$

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$$\mathcal{M}^\dagger{}_{x,y} = A\mathcal{M}_{x,y}A,$$ for Wilson $A = \gamma_5, \mu = \{0, i\hat{\mu}\}$ with $\hat{\mu} \in \mathcal{R}$

$$< \mathcal{O} > = \int \mathcal{D}U\, \mathcal{O}[U] e^{i\psi}\, e^{-S_G - S_{eff}} / \int \mathcal{D}U\, e^{i\psi}\, e^{-S_G - S_{eff}}$$

Lattice QCD thermodynamics

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$$T = (N_\tau a_\tau(\beta))^{-1}, \qquad V = (N_\sigma a_\sigma(\beta))^3,$$

$$\frac{\epsilon - 3p}{T^4} = -\frac{1}{VT^3}\left(a\frac{\partial\beta}{\partial a}\frac{\partial \ln \mathcal{Z}}{\partial\beta} + a\frac{\partial m}{\partial a}\frac{\partial \ln \mathcal{Z}}{\partial m}\right), \qquad -pV = \frac{T}{V}\ln\mathcal{Z} = E - T\mathcal{S} - \mu_q n_q$$

From the Euclidean action $S(\beta, m, \mu)$

$$a\frac{dS}{da} = 3V\frac{\partial S}{\partial V} - T\frac{\partial S}{\partial T}$$

$$V\frac{\partial\Omega}{\partial V} = VT\left\langle\frac{\partial S}{\partial V}\right\rangle = -pV$$

$$T\frac{\partial\Omega}{\partial T} = \Omega + T^2\left\langle\frac{\partial S}{\partial T}\right\rangle = -T\mathcal{S} = \Omega - E + \mu_q N_q$$

$$\frac{T}{V}\left\langle a\frac{dS}{da}\right\rangle = \epsilon - 3p - \mu_q n_q = -\frac{T}{V}\left(a\frac{\partial\beta}{\partial a}\frac{\partial \ln \mathcal{Z}}{\partial\beta} + a\frac{\partial m}{\partial a}\frac{\partial \ln \mathcal{Z}}{\partial m} + a\frac{\partial\mu}{\partial a}\frac{\partial \ln \mathcal{Z}}{\partial\mu}\right)$$

Taylor expansion about $\mu = \mu_q a = 0$ leads to

$$\Delta\left(\frac{p}{T^4}(\mu)\right) = \frac{p}{T^4}\bigg|_{T,\mu_q} - \frac{p}{T^4}\bigg|_{T,0} = \frac{1}{2!}\frac{N_\tau^3}{N_\sigma^3}\mu^2\frac{\partial^2 \ln\mathcal{Z}}{\partial\mu^2} + \frac{1}{4!}\frac{N_\tau^3}{N_\sigma^3}\mu^4\frac{\partial^4 \ln\mathcal{Z}}{\partial\mu^4} + \cdots$$

$$= \sum_{p=1}^{\infty} c_p(T)\left(\frac{\mu}{T}\right)^p$$

Simulating QCD at $\mu \neq 0$

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  Derivatives of screening mass and $< \bar{q}q >$ [Choe,*et al.* PRD65:054501]
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- **Glasgow:** Zero in complex μ-plane, Lee-Yang zero [Barbour, NP60A:229]

$$Z = \frac{\int \mathcal{D}U \det \mathcal{M}[\mu] \exp(-\beta S_G)}{\int \mathcal{D}U \det \mathcal{M}[0] \exp(-\beta S_G)} = \left\langle \frac{\det \mathcal{M}[\mu]}{\det \mathcal{M}[0]} \right\rangle_{\mu=0} = \sum_{n=-3N_\sigma^3}^{+3N_\sigma^3} < b_n > e^{n\mu}/T$$

It is difficult to obtain $< b_n >_{\mu=0}$ numerically at low T when μ increases

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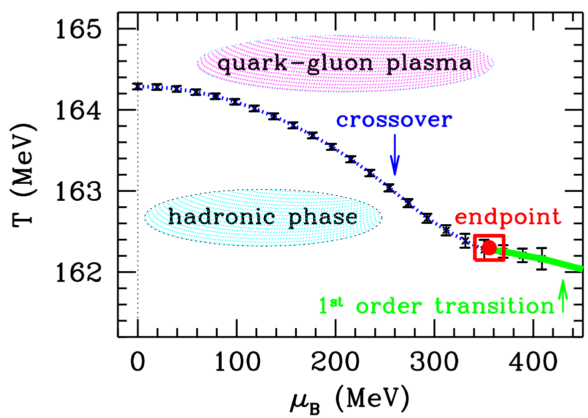
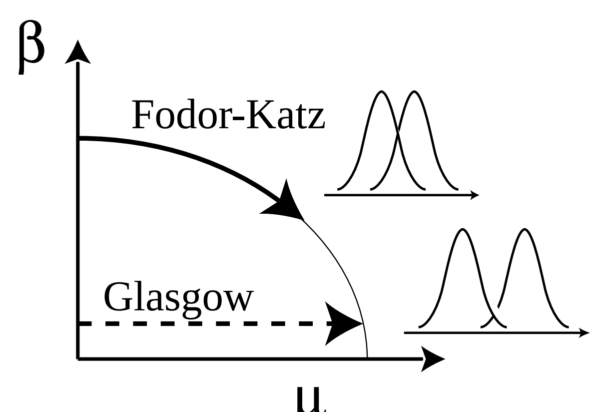
- **Multi-parameter reweighting method:** Transition line at $\mu \neq 0$ [Fodor and Katz JHEP0203:014]

$$< \mathcal{O} > = \frac{\int \mathcal{D}U \mathcal{O} \frac{\det \mathcal{M}[\mu]}{\det \mathcal{M}[0]} e^{-(\beta-\beta_0)S_G} \det \mathcal{M}[0] e^{-\beta_0 S_G}}{\mathcal{Z}(\mu)} = \frac{\left\langle \mathcal{O} \frac{\det \mathcal{M}[\mu]}{\det \mathcal{M}[0]} e^{-\Delta\beta S_G} \right\rangle_0}{\left\langle \frac{\det \mathcal{M}[\mu]}{\det \mathcal{M}[0]} e^{-\Delta\beta S_G} \right\rangle_0}$$





Taylor expansion at $\mu \neq 0$

Bielefeld-Swansea: Derivatives with respect to $\mu = 0$ [Alton, *et al*, PRD66:074507]

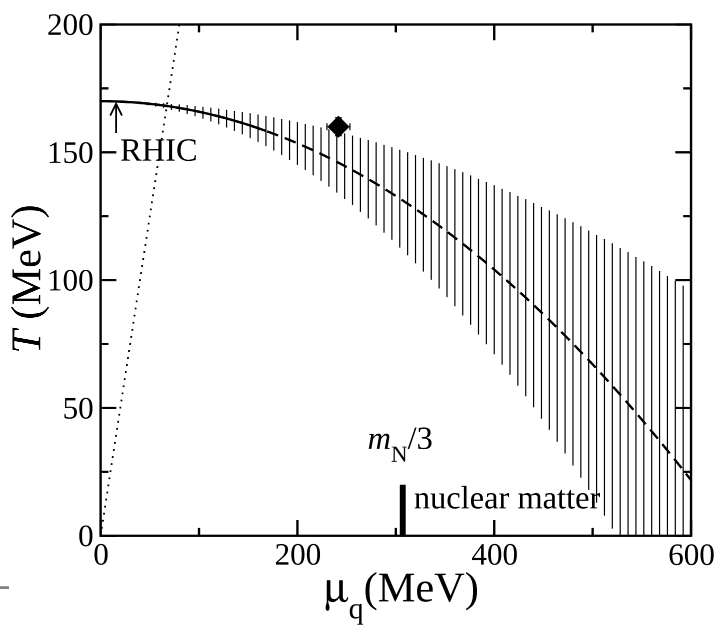
Reweighting for the gauge and fermion parts of Wilson action read, respectively

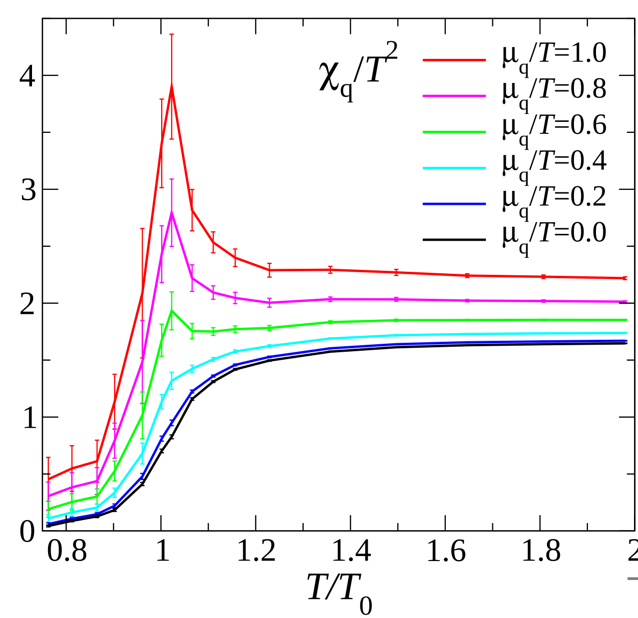
$$S_G(\beta) - S_G(\beta_0) = (\beta - \beta_0) \sum_{x,\mu>\nu} P_{\mu\nu}(x), \qquad \text{plaquette}$$

$$\ln \left(\frac{\det \mathcal{M}[\mu]}{\det \mathcal{M}[0]} \right) = \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n \ln \det \mathcal{M}}{\partial \mu^n} \equiv \sum_{n=1}^{\infty} \mathcal{R}_c \mu^n$$

It is easier to calculate the phase μ Im Tr$\mathcal{M}^{-1} \frac{\partial \mathcal{M}}{\partial \mu}$ than the determinant itself.

Expand the fermionic observables such as chiral condensate $< \bar{\psi}\psi >= \partial \ln \mathcal{Z} / \partial m_q = c <$ Tr $\mathcal{M}^{-1} >$ with the identity $\frac{\partial \mathcal{M}^{-1}}{\partial x} = -\mathcal{M} \frac{\partial \mathcal{M}}{\partial x} \mathcal{M}^{-1}$ one can get expressions for $\frac{\partial^n \ln \det \mathcal{M}}{\partial \mu^n}$ and $\frac{\partial^n \text{Tr } \mathcal{M}^{-1}}{\partial \mu^n}$.





Thermodynamical quantities at $\mu \neq 0$

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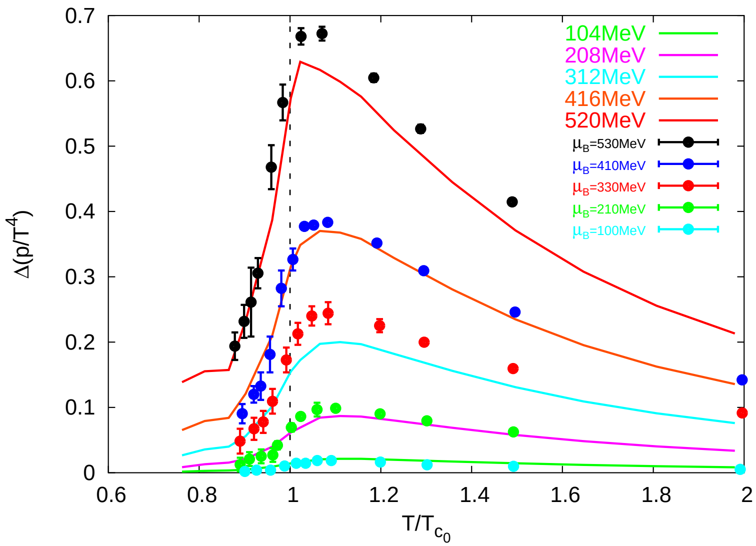
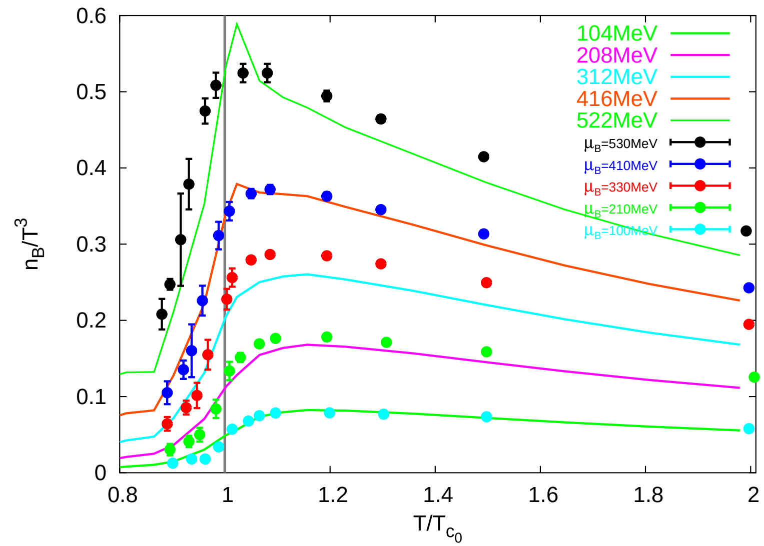




Fofor and Katz: hep-lat/0208078 $n_f = 2+1$ standard staggered action on $\{8, 10, 12\}^3 \times 4$ with $m_{ud} \sim 65$, $m_s \sim 135$ MeV and $m_\pi/m_\rho \sim 0.66$. Results multiplied with correction factor taking into account the descritization and the continuum limits (SB).

Bielefeld-Swansea: PRD68:014507 $n_f = 2$ p_4 improved staggered fermions on $10^3 \times 4$ with mass $m/T = 0.4$

Imaginary μ

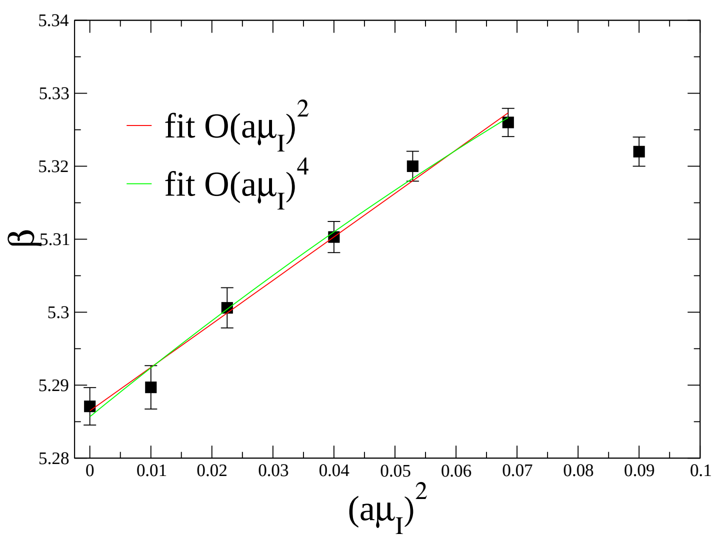
Imaginary μ: Pure imaginary $\mu \rightarrow i\mu_I$ [de Forcrand, Philipsen, NPB642:290]

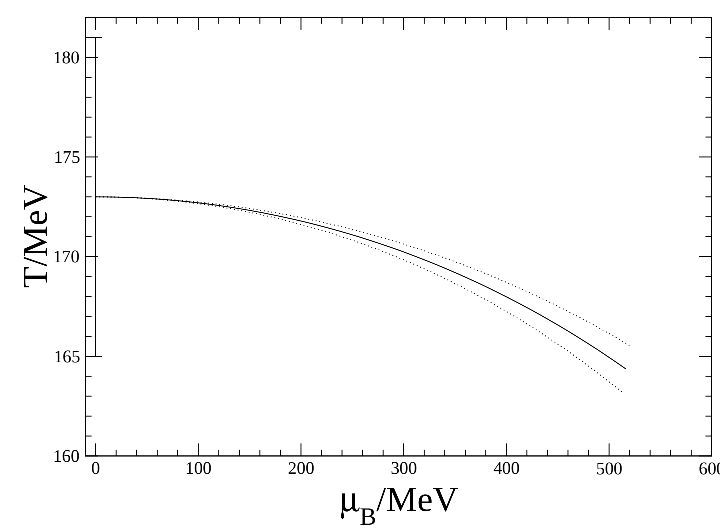
$$\mathcal{M} = \mathcal{D}_\nu \gamma_\nu + m + i\mu_I \gamma_0, \qquad \Longrightarrow \qquad \mathcal{M}^\dagger = \gamma_5 \mathcal{M} \gamma_5$$

The connection to real chemical potential is provided by

$$\mathcal{Z}(T, m_q) = \int_{-\pi T}^{+\pi T} \frac{d\mu_I}{2\pi T} \mathcal{Z}(T, i\mu_I) e^{-i\mu_I m_q / T}$$

Using the analyticity of the partition function to continue expectation values computed with





Resonance gas model

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Collaborators: Frithjof Karsch and Krzysztof Redlich

$$\mathcal{Z}(T,V)|_{\mu=0} = \mathrm{Tr}\left[e^{-\beta H}\right], \qquad \ln \mathcal{Z}(T,V)|_{\mu=0} = \sum_i \ln \mathcal{Z}^{(1)}(T,V)\Big|_{\mu=0}$$

$$\ln \mathcal{Z}^{(1)}(T,V)\Big|_{\mu=0} = V\frac{g_i}{2\pi^2}\int_0^\infty dk\; k^2\; \eta \ln(1+\eta e^{-\beta E_i})$$

$$\epsilon = -\frac{1}{V}\beta\frac{\partial \ln \mathcal{Z}}{\partial \beta} = \sum_i \epsilon_i^{(1)}, \qquad \frac{\epsilon^{(1)}}{T^4} = \frac{g_i}{2\pi^2}\sum_{j=1}^{\infty}(-\eta)^{j+1}\frac{(\beta m_i)^3}{j}\left[3\frac{K_2(j\beta m_i)}{j\beta m_i} + K_1(j\beta m_i)\right]$$

$$\frac{\epsilon^{(1)} - 3p^{(1)}}{T^4} = \frac{g_i}{2\pi^2}\sum_{j=1}^{\infty}(-\eta)^{j+1}\frac{(\beta m_i)^3}{j}K_1(j\beta m_i)$$

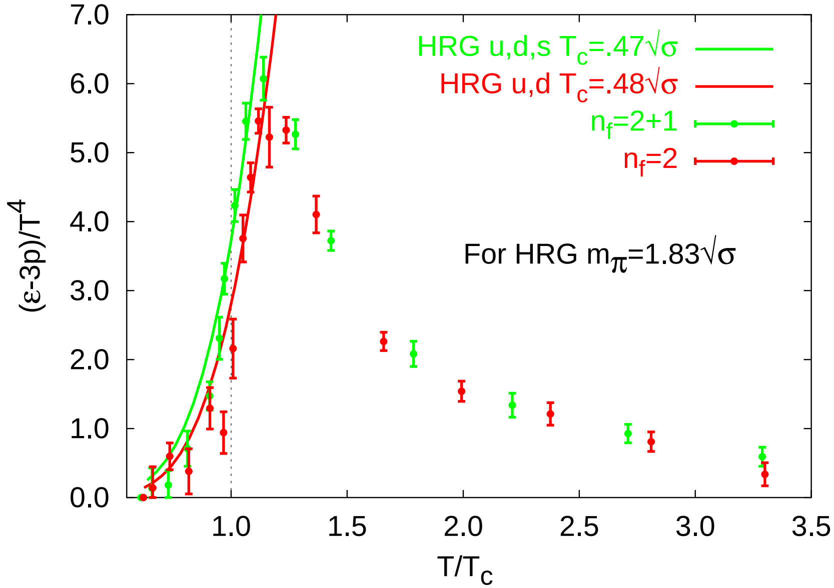
ϵ starts rising rapidly at $T\sim 160$ MeV. It reaches 0.3 GeV$/$fm^3 at $T\sim 155$ MeV and 1.0 GeV$/$fm^3 at $T\sim 180$ MeV. On the lattice $\epsilon\sim 0.7$ GeV$/$fm^3 at $T\sim 170$ MeV. The change in ϵ with different n_f is accompanied by a shift in $T_c \longrightarrow$ Percolation

At $T\sim 170$ MeV, a simple pion gas gives $\epsilon\sim 0.1$ GeV$/$fm^3!

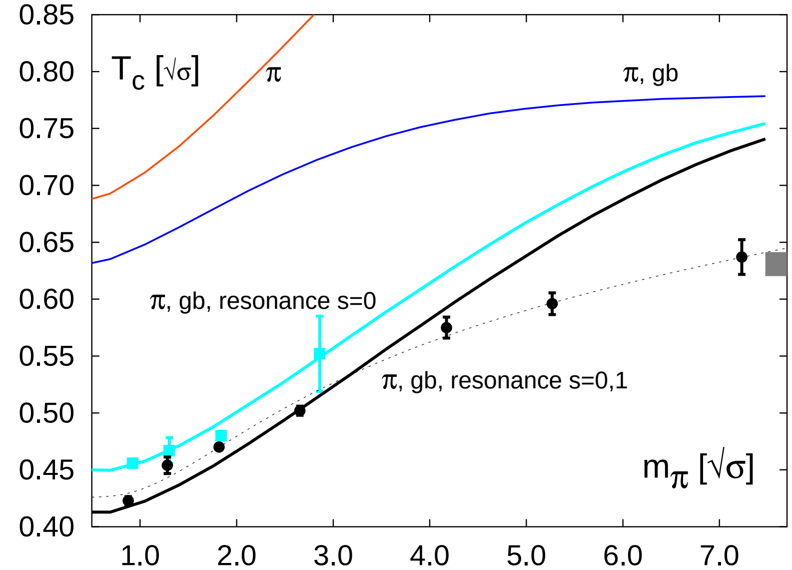
On the lattice it has been found that for very small m_q there is a *true* phase transition. For intermediate m_q the transition is NOT related to any singularity. Only rapid change in thermodynamical quantities in a narrow T-interval is realized.

Check the dependence of m_q on the critical temperature.

Resonance gas model at $\mu = 0$



$n_f = 2+1$ with p_4 action



$n_f = 2, n_f = 3$

$$\frac{\epsilon_{SB}}{T^4} = \left(8 + \frac{21}{4} g_{eff}\right) \frac{\pi^2}{15},$$

$$\frac{p_{SB}}{T^4} = \left(8 + \frac{21}{4} g_{eff}\right) \frac{\pi^2}{45}$$

$$\left(\frac{T_c}{\sqrt{\sigma}}\right)_{m_{PS}/\sqrt{\sigma}} = 0.4 + 0.04(1) \left(\frac{m_{PS}}{\sqrt{\sigma}}\right)$$

Resonance gas model at $\mu \neq 0$

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$$\frac{p}{T^4} = \sum_i \frac{g}{\pi^2} \left(\frac{m_i}{T}\right)^2 K2\left(\frac{m_i}{T}\right) \cosh\left(\frac{\mu_b}{T}\right)$$

Thus that total baryonic contribution to the pressure in resonance gas is

$$\frac{p_B}{T^4} = F(T) \cosh\left(\frac{\mu_b}{T}\right)$$

In the Boltzmann approximation we have

$$\frac{\Delta p}{T^4} = F(T) \left[\cosh\left(\frac{\mu_b}{T}\right) - 1\right] \simeq F(T) \left(\tilde{c}_2 \left(\frac{\mu_q}{T}\right)^2 + \tilde{c}_4 \left(\frac{\mu_q}{T}\right)^4\right)$$

$$\frac{n_q}{T^3} = 3F(T) \sinh\left(\frac{\mu_b}{T}\right) \simeq F(T) \left(2\tilde{c}_2 \left(\frac{\mu_q}{T}\right) + 4\tilde{c}_4 \left(\frac{\mu_q}{T}\right)^3\right)$$

$$\frac{\chi_q}{T^2} = 9F(T) \cosh\left(\frac{\mu_b}{T}\right) \simeq F(T) \left(2\tilde{c}_2 + 12\tilde{c}_2 \left(\frac{\mu_q}{T}\right)^3\right)$$

$\tilde{c}_2 = 9/2$, $\tilde{c}_4 = 27/8$ and $\tilde{c}_6 = 729/720$. The expansion coefficients $c_{2n} = \tilde{c}_{2n} F(T)$

For fixed μ_q/T the ration of expansion coefficients are T-independent.

Pressure and susceptibility

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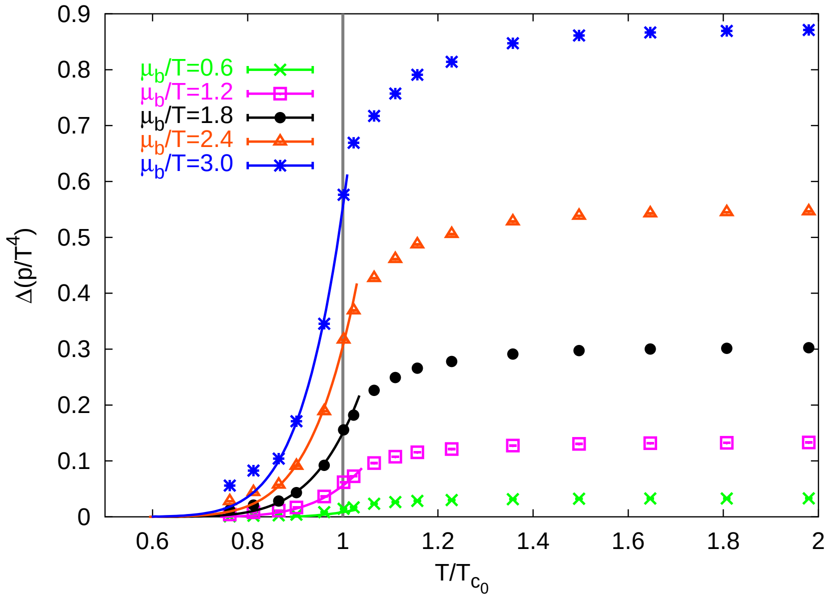
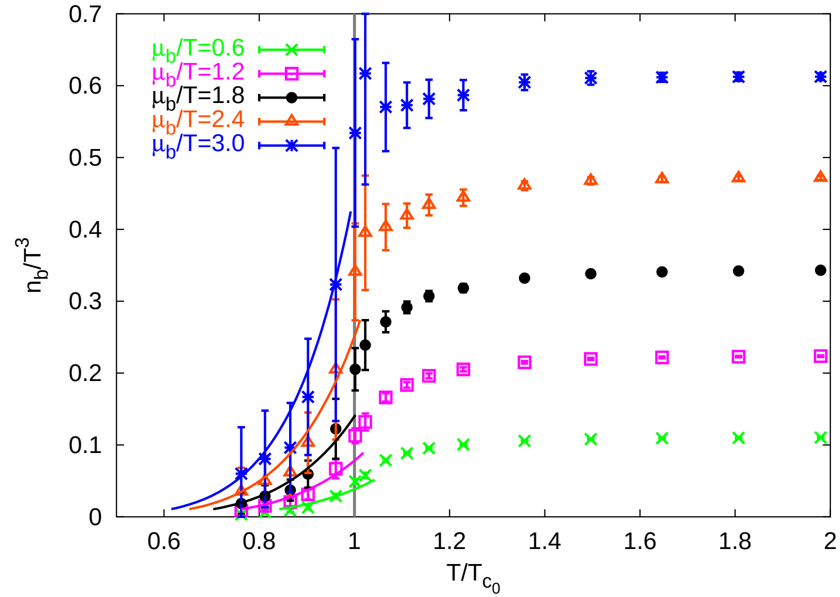




Non-truncated expressions

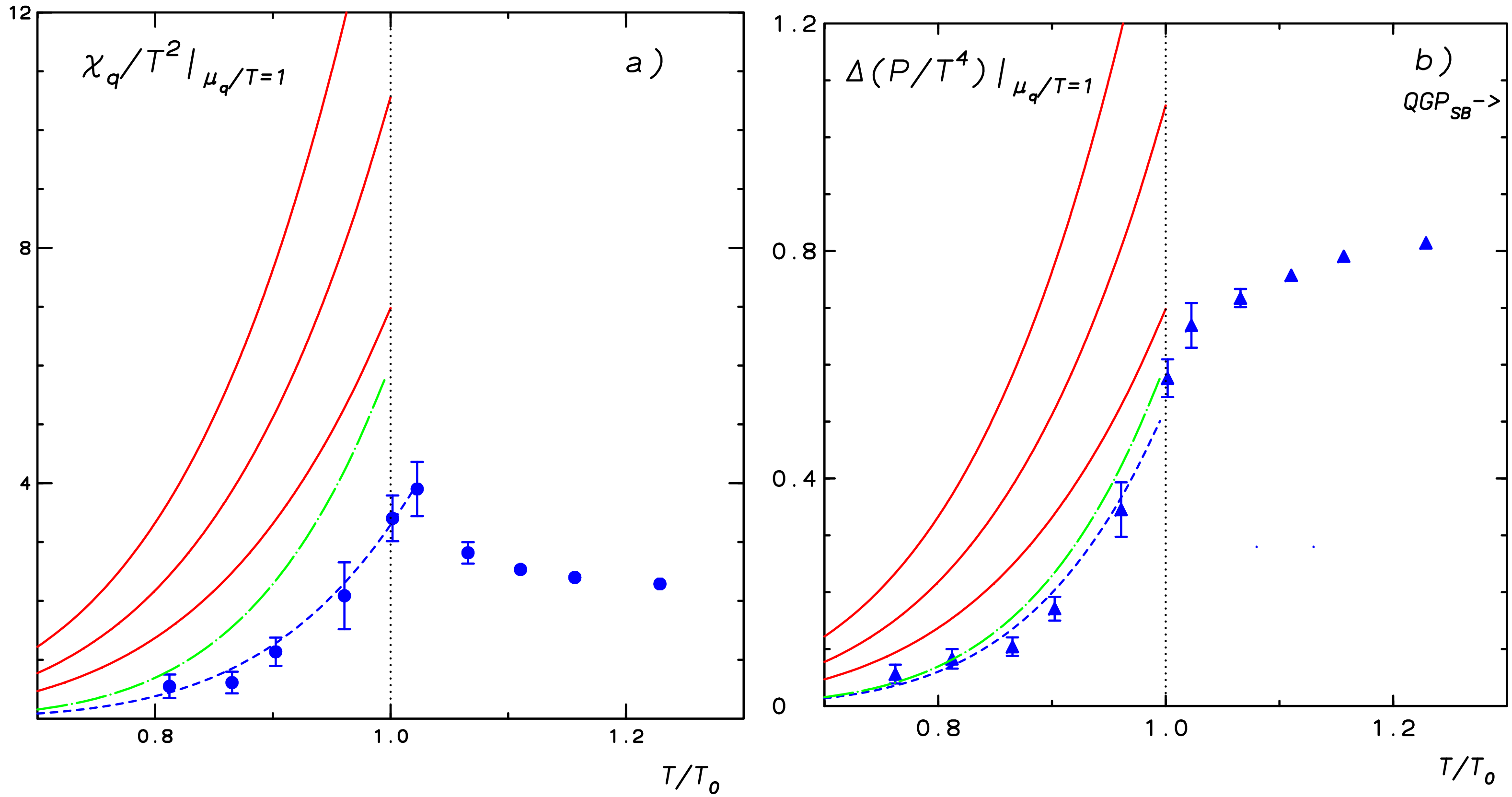
$\mu_b = 3\mu_q$ $\mu_{PS} \sim 800$ MeV $T_0 \sim 200$ MeV

For $T > T_c$,

Quasi-particles [B. Kämpfer, *et al.*]

Resonance width, lifetime, etc. [K. Bugaev, D. Blaschke]

Transition temperature vs. hadron mass



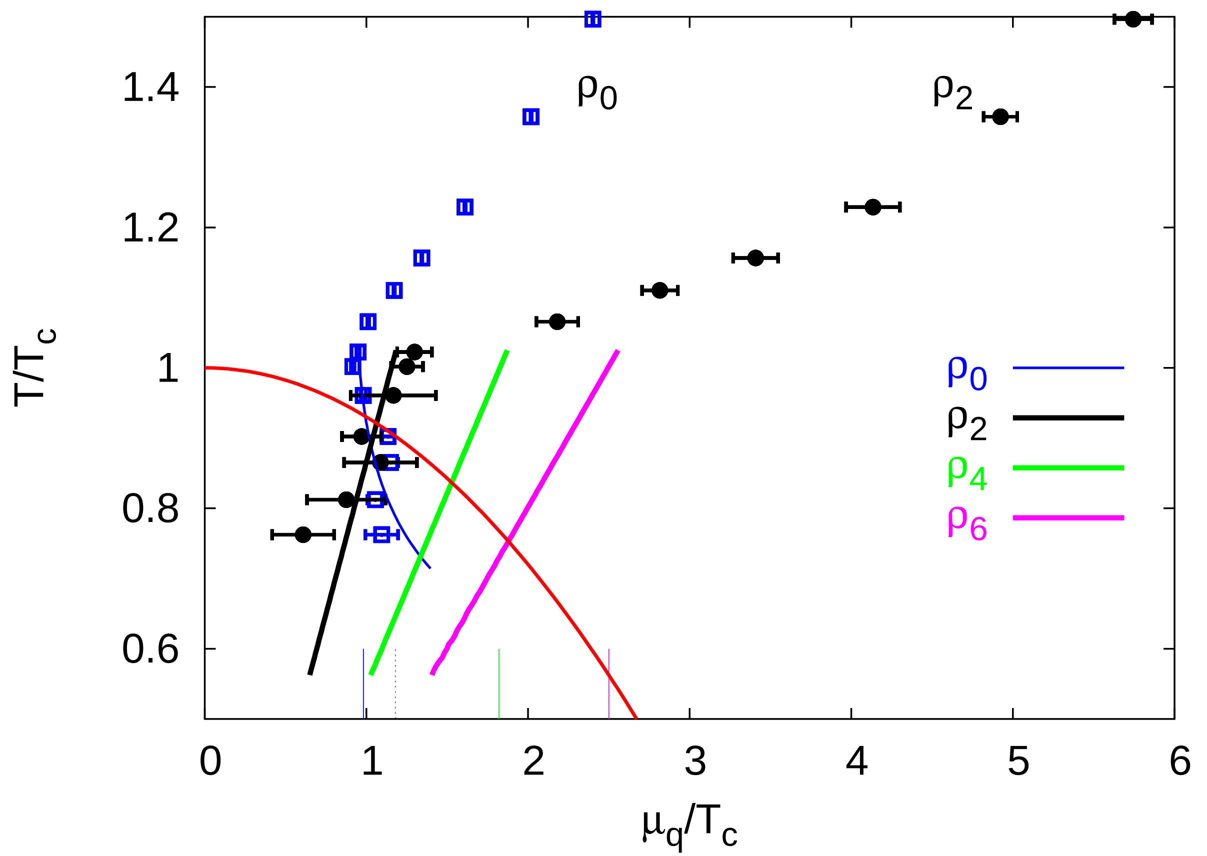
Dashed line: $m_\pi = 780$ MeV, $\mathcal{O}((\mu_q/T)^4)$

Dashed-dotted line: $m_\pi = 780$ MeV, no truncation

Full line: physical masses complete expression and $T_c = 160, 170, 180$ MeV (bottom to top)

p/T^4 increases when $m_h(m_\pi) \to m_h^{phys}$. But it will be reduced since T_c decreases.

Radius of conversion and T_c



$$\rho = \lim_{n\to\infty} \rho_n \equiv \lim_{n\to\infty} \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|}, \qquad (\to \infty \text{in HRG})$$

$$c_0 = p_M + F_M(T) \longrightarrow \rho_0 = \sqrt{\left|\frac{p_M + F_M(T)}{\frac{9}{2}F_b(T,\mu=0)}\right|}$$

Summary

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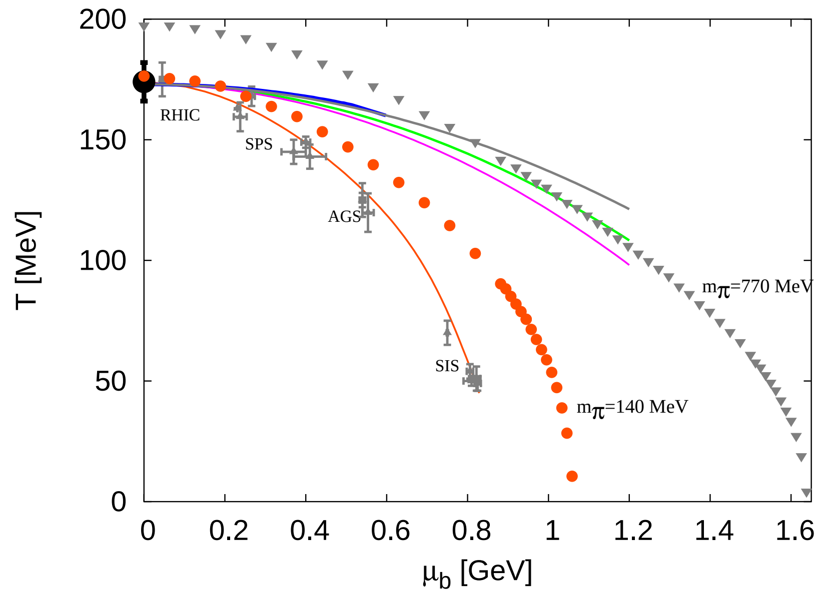

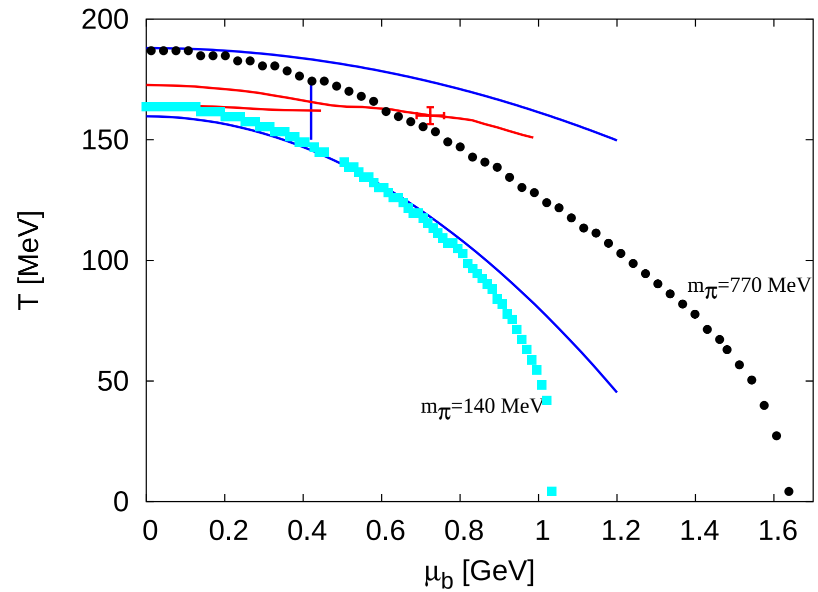

- At $\mu = 0$ the lattice calculations are using very heavy quark masses, and
- at $\mu \neq 0$ the Taylor-expansions have to be truncated.
- The resonances are the essential degrees of freedom near the transition point.
- The partition function of resonance gas gives a consistent description of QCD equation of state.
- QCD transition can be given by the condition of fixed energy density