# Large Extra Dimensions and the Minimal Length


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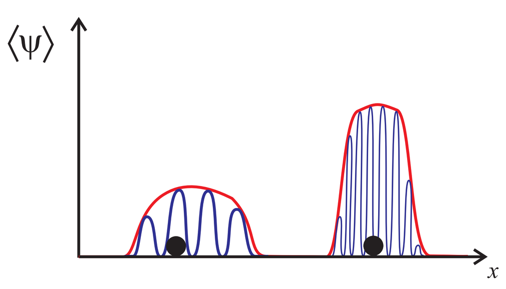
## The Minimal Length Scale

It was in the 5th century b.c. that Demokrit postulated a smallest particle out of which matter is build. He called it an "atom". In Greek, the prefix "a" means "not" and the word "tomos" means cut. Thus, atoms or atom means uncuttable or undividable.
2500 years later, we know that not only the atom is dividable, but also is the atomic nucleus. The nucleus is itself a composite of neutrons and protons and further progressions in science have revealed that even the neutrons and protons have a substructure. Is there an end to this or will the quarks and gluons turn out to be non-fundamental too?
An answer to this can not be given yet, but judging from one of the most promising candidates for an unified theory – String Theory – there is indeed a smallest possible resolution of spacetime. This is not surprising since the success of String Theory is due to the very reason that the extension of strings is finite. The scale for this expected minimal length is given by the string scale which is close to the Planck-scale $l_{\rm p}$.

D.J.Gross [hep-th/9704139]: "In String Theory [...] the probes themselves are not pointlike but rather extended objects, and thus there is another limitation as to how precisely we can measure short distances. As energy is pumped into the string it expands and thus there is an additional uncertainty proportional to the energy"

The occurrence of this minimal length scale has to be expected from very general reasons, not only in String Theory but in all theories at high energies.

Usually, every sample under investigation can be resolved by using beams of an energy high enough to assure the Compton-wavelength $\lambda$ is below the size of the probe. The smaller the sample, the higher the energy must become and thus, the bigger the collider.

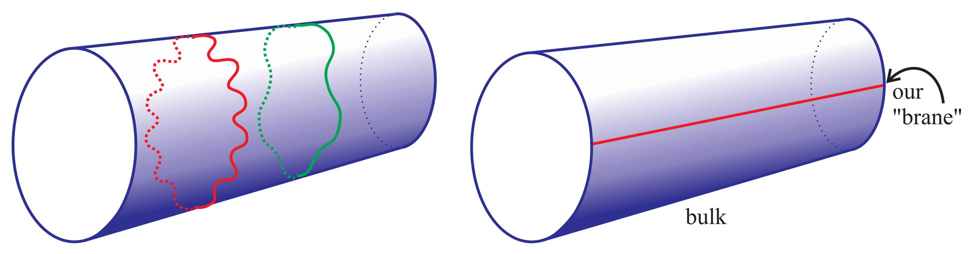



When the size of the probe should be as small as the Planck-length, the energy needed for the beam would be about Planck-mass. The Planck-mass is the mass at which contributions of quantum gravity get important. The pertubation of space-time causes an uncertainty in addition to the usual uncertainty in quantum mechanics. This additional uncertainty increases with energy and makes it impossible to probe distances below the Planck-length.
The Planck-length which is derived from the Standard Model of physics is $\approx 10^{20}$ fm and thus far out of reach for future experiments. But this depressing fact looks completely different if we work within the model of Large Extra Dimensions (LXDs).

## Large Extra Dimensions

During the last decade several models using compactified LXDs as an additional assumption to the quantum field theories of the Standard Model have been proposed. These effective models are motivated by String Theory and provide us with an useful description to predict first effects beyond the standard model. They do not claim to be a theory of first principle or a candidate for a grand unification. Instead they allow us to compute testable results which can in turn help us to gain insights about the underlying theory.
There are different ways to build a model of extra dimensional space-time:

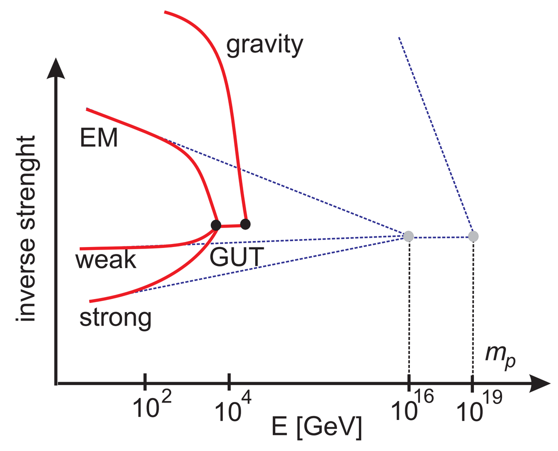
- The ADD -model proposed by Arkani-Hamed, Dimopoulos and Dvali adds $d$ extra spacelike dimensions without curvature, in general each of them compactified to the same radius $R$. All Standard Model particles are confined to our brane, while gravitons are allowed to propagate freely in the bulk.
- The setting of the model from Randall and Sundrum is a 5-dimensional spacetime with an non-factorizable geometry. The solution for the metric is found by analyzing the solution of Einsteins field equations with an energy density on our brane, where the Standard Model particles live.
- Within the model of universal extra dimensions all particles (or in some extensions, only bosons) can propagate in the whole multi-dimensional spacetime. The extra dimensions are compactified on an orbifold to reproduce standard model gauge degrees of freedom.




Due to the compactification, momenta in the direction of the LXDs can only occur in quantized steps $\propto 1/R$. This yields an infinite number of equally spaced excitations, the so called Kaluza-Klein-Tower.
The existence of LXDs leads to a theoretical prescription which lowers the scale of quantum gravity down to values comparable to the scales of the Standard Model interactions. We will denote this new fundamental scale by $M_f$.

In order to solve the hierarchy problem $M_f$ should be in the range of $\approx$ TeV. In the models with LXDs, first effects of quantum gravity occur at energies $\approx M_f$. They would be observable in soon future at the LHC. Some of the predicted effects are:

- Production of gravitons.
- Modifications due to virtual graviton exchange.
- Production of black holes.

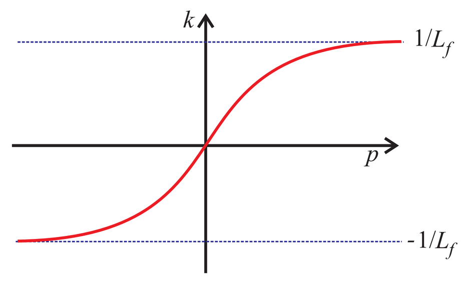



The figure shows schematically the renormalisation group equations with (red) and without (blue) LXDs.

For self-consistence, the models further have to consider the fact that a lowering of the fundamental scale leads to a raising of the minimal length. Within the model of LXDs not only effects of quantum gravity occur at lowered energies but so do the effects of the minimal length scale.

## Building a Model for the Minimal Length

The key idea for a model which includes the minimal length is that the effects from a finite resolution are similar to a finite minimal Compton wavelength of particles or a maximal frequency, respectively. Usually, the relation between the momentum $p$ and the wavevector $k$ of the particle is linear: $p = \hbar k$, $E = \hbar\omega$. Now, we set the wave vector to be a function of momentum $k = k(p)$, $\omega = \omega(E)$ with an upper bound as pictured in the plot below (e.g. with a tanh).




This new relation has the effect that the energy of a particle can be increased arbitrarily, but its wavelength can not decrease below a new scale $L_f$ that is the new Planck-scale in our model.

## The Minimal Length in Quantum Mechanics

This approach can now be incorporated into quantum mechanics. We quantize via the commutation relations and add the new functional relation between wavevector and momentum to the quantisation procedure [1]:

$$[\hat{x}_i, \hat{k}_j] = {\rm i}\delta_{ij} \quad , \hat{p} = \hat{k}(p) \quad \Rightarrow [\hat{x}, \hat{p}] = +{\rm i}\frac{\partial p}{\partial k} \quad .$$

With an expansion ot the tanh, $k(p) \approx \frac{p}{M_f} - \left(\frac{p}{M_f}\right)^3$, we get

$$[\hat{x}, \hat{p}] \approx {\rm i}\,\hbar\left(1 + \frac{\hat{p}^2}{M_f^2}\right) \quad , \quad \Delta p \Delta x \geq \frac{1}{2}\hbar\left(1 + \frac{\langle \hat{p}^2 \rangle}{M_f^2}\right) \quad .$$

So, we find a generalized uncertainty principle! And further, the measure in momentum space is modified

$$\langle p' | p \rangle == \frac{\partial p}{\partial k}\delta\left(p - p'\right) \quad , \quad {\rm d}p \to \frac{{\rm d}p}{\hbar}\frac{\partial k}{\partial p} \quad .$$

With use of an expansion of the tanh for high energies we have $\partial p/\partial k \approx \exp\left(-|p|/M_f\right)$ and so we can draw the important conclusion, that the momentum measure is exponentially squeezed at high energies!
The quantization of the energy-momentum relation yields the modified Klein-Gordon and the Dirac Equation

$$\eta^{\mu\nu}\hat{p}_\nu(k)\hat{p}_\mu(k)\psi = m^2\psi \quad , \quad (\not{p}(k) - m)\psi = 0$$

where $p$ is now a function of $k$.

## Effects of the Minimal Length Scale

These relations lead to modifications of the familiar equations in quantum mechanics and can be used to make predictions for effects that should arise from the existence of a minimal length scale. The momentum operator in position representation can be derived and with this, the modified Schrödinger equation

$$\tilde{p} = -{\rm i}\,\hbar\nabla\left(1 - L_f^2\Delta\right) \quad , \quad \tilde{H} = -\frac{\hbar^2}{2m}\left[\nabla\left(1 - L_f^2\Delta\right)\right]^2 + V(r) \quad .$$

### The Hydrogen Atom

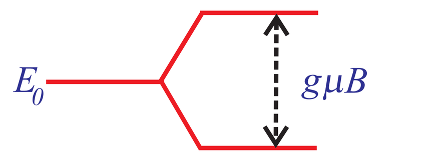
For the hydrogen atom we insert $V(r) = e^2/r$ and find a shift of the energy levels from the old values $E_n$ to the new values $\tilde{E}_n$. With the current accuracy of experiments for the Hydrogen S1-S2 transition, we find a very weak constraint on the new scale.

$$\tilde{E}_n \approx E_n\left(1 - \frac{4}{3}\frac{m_e E_0}{M_f^2 n^2}\right) \quad \Rightarrow \quad M_f \approx> 10 \text{ GeV} \quad .$$

The results with the new model are so far in agreement with other minimal length formalism "on the market": A. Kempf $et\ al$, F. Brau $et\ al$, I. Dadic $et\ al$ etc.

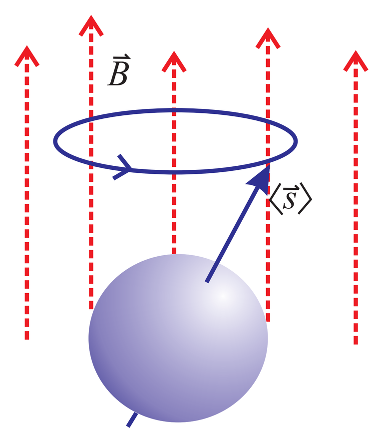
### The Muon $g-2$

To derive useful predictions we have to look at high energy experiments or high precision observables, such as the magnetic moment of the muon $g-2$ [2].




A particles energy in a magnetic field $B$ depends on its spin. Energy levels that are degenerated for free particles split up in the presence of a field.

In a static, homogenous magnetic field, the expectation value of the spin vector will perform a precession around the direction of the field. The rotation frequency is proportional to the strength of the field and the magnetic moment of the particle and so can be used to measure the magnetic moment.




The value of $g$ is modified by selfenergy corrections in quantum field theory. Modifications from the minimal length should be important even at the classical level and occur in the QED-range.
The experimental value for the magnetic moment of the muon is known today to extremely high precission. The modifications arising from the minimal length can be derived by coupling the particle to the electromagnetic field $K_\nu \to k_\nu - eA_\nu$ in the usual way

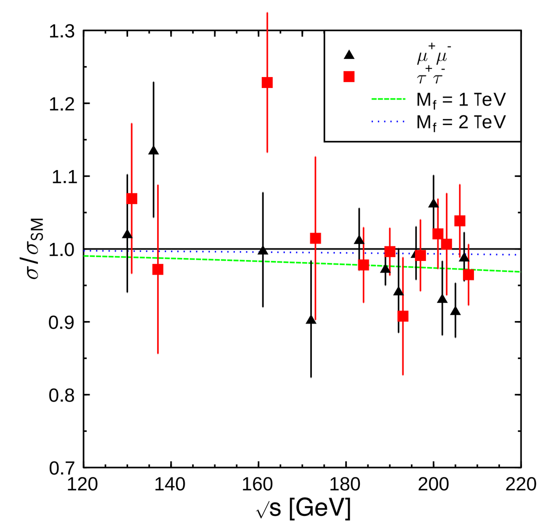
$$\omega|\psi\rangle \approx \gamma^0\left(\gamma^i\hat{K}_i + \frac{m}{\hbar}\right)\left(1 - \frac{\hbar^2\hat{K}^i\hat{K}_i + m^2}{M_f^2}\right)|\psi\rangle$$

Analysing this equation, one finds a constraint on the new scale in the range of the constraints from LXDs: $M_f \approx 0.67$ TeV.

### Tree Level Processes

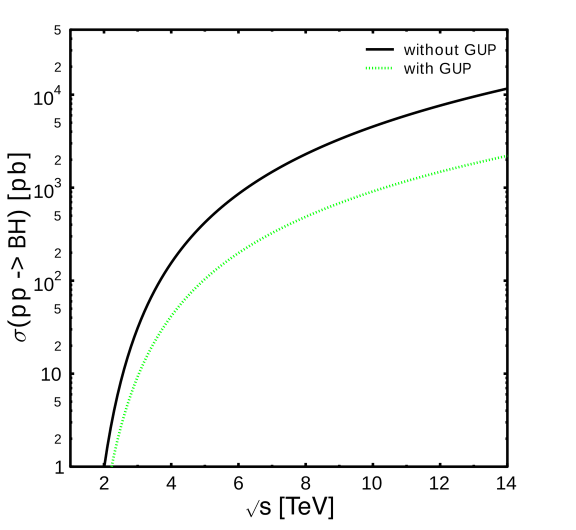
To go for the high energy observables, we examine how the minimal length influences cross-sections. We compute the modifications of $A + B \to X + Y$ cross-sections for QED at tree level and find that an extra factor arises! The dominant contribution is the sqeezing-factor from the measure of momentum space which lowers the phase space of the outgoing particles. This yields a cross-section that is below the Standard Model value. Applying this result to fermionic pair production processes $e^+e^- \to \mu^+\mu^-$ or $e^+e^- \to \tau^+\tau^-$, resp., we find that the modified cross-sections are in agreement with the data for a minimal length in the range $L_f \approx 10^{-4}$ fm.

The plot shows the energy dependence of the ratio from the new total cross-section value to the Standard Model cross-section for fermionic pair production for different values of the minimal length scale. The data points are taken from LEP2, D. Bourikov $et\ al.$, LEP2FF/01-02 (2001).




### And More

The effect of the sqeezed momentum space does also modify predictions from the LXD-scenario. Since the modifications get important at energies close to the new scale, the predicted graviton and black hole production is strongly influenced.




Black hole production is less propable[3] since the approach of the partons to distances below the Schwarzschild-radius needs higher energies within the minimal length scenario. The plot shows the total cross-section of the black hole production. For LHC com energies $\sqrt{s} \approx 14$ TeV the rate is lowered by a factor $\approx 5$.

Furthermore, the minimal length acts as a natural regulator[4] for ultraviolet divergences! It therefore reliably removes an inherent arbitrariness of choice for of cut-off regulators in higher dimensional quantum field theories.

## Publications


[1] S. Hossenfelder, M. Bleicher, S. Hofmann, J. Ruppert, S. Scherer and H. Stocker, "Collider signatures in the Planck regime", Phys. Lett. B **575**, 85 (2003), [arXiv:hep-th/0305262].

[2] U. Harbach, S. Hossenfelder, M. Bleicher and H. Stocker, "Probing the minimal length scale by precision tests of the muon g-2", Phys. Lett. B **584**, 109 (2004), [arXiv:hep-ph/0308138].

[3] S. Hossenfelder, "Suppressed black hole production from minimal length", [arXiv:hep-th/0404232].

[4] S. Hossenfelder, "Running coupling with minimal length", [arXiv:hep-ph/0405127].