The Longitudinal Structure Function at Third Order

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in collaboration with **J.A.M. Vermaseren** and **A. Vogt**, hep-ph/0411112

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Plan

- Motivation
- Method and Results
- Summary

Motivation

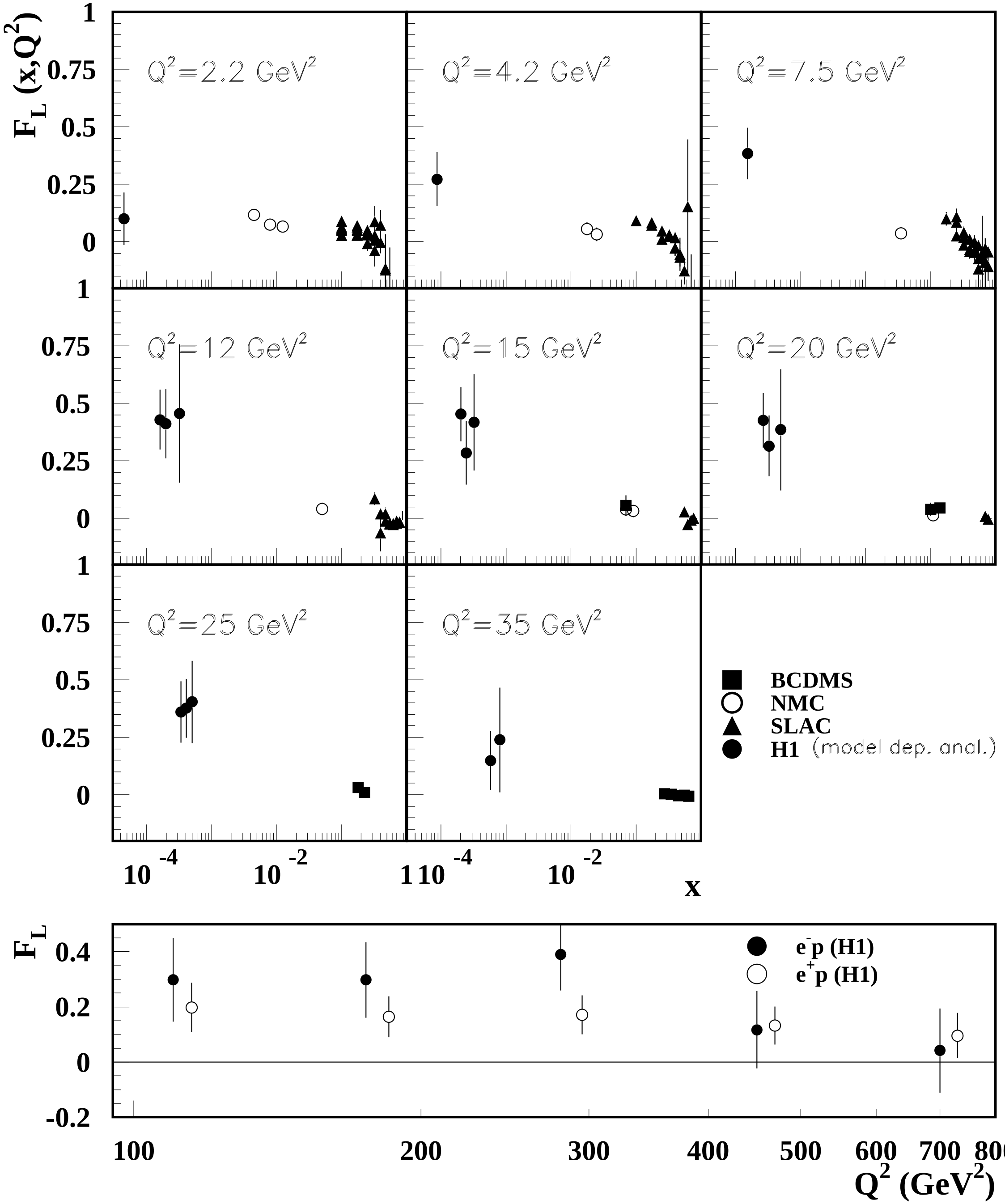
Some facts about F_L

- Theory
  - Longitudinal structure function $F_L = F_2 - 2xF_1$ vanishes at tree level
  - Complete NNLO analysis of F_L (or $R = \frac{\sigma_T}{\sigma_L}$) requires three-loop coefficient functions
- Experiment
  - Various methods for measurement of F_L
    - R-ratio
    - derivative $\frac{\partial F_L}{\partial \ln y}$ (inelasticity y)
    - fits of the shape of F_L in cross section
  - HERA results for F_L

    H1 collaboration ‘01; H1 collaboration ‘03; Klein ‘03; Lobodzinska ‘03

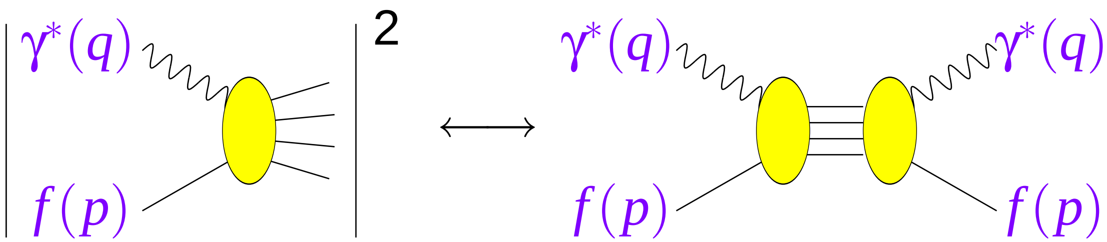
Data

- World data for F_L
  PDG 2004



Method and Results

Optical theorem



- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}F_3(x,Q^2)$$

- Longitudinal structure function F_L

$$x^{-1}F_L \;=\; C_{L,\mathrm{ns}} \otimes q_{\mathrm{ns}} + \langle e^2 \rangle \left(C_{L,\mathrm{q}} \otimes q_{\mathrm{s}} + C_{L,\mathrm{g}} \otimes g \right)$$

  - number distributions of quarks and gluons q_i, g

The calculation (in a nut shell)

- Calculate coefficient functions $c_{2,\mathrm{a}}$ and $c_{L,\mathrm{a}}$ (Mellin moments) from Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

$$c_{i,\mathrm{a}}^{(n)}(N) \; = \; \int_0^1 dx \; x^{\,N-1} \, c_{i,\mathrm{a}}^{(n)}(x) \, , \qquad i = 2, 3, L$$

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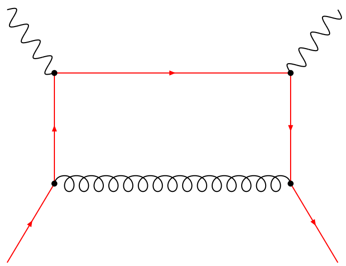
- **One-loop** Feynman diagrams

  $\longrightarrow$ in total 18 for $c_{i,\mathrm{a}}^{(1)}, i = 2, 3, L$

  (pencil + paper)



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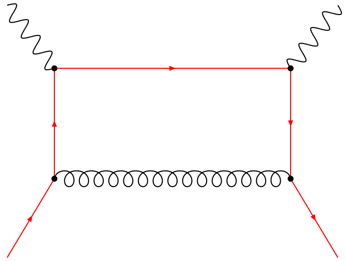
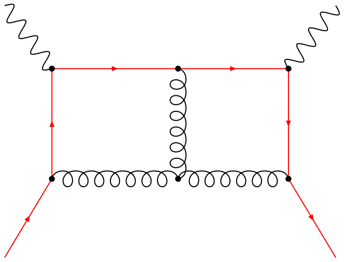


- **Two-loop** Feynman diagrams
  $\longrightarrow$ in total 350 for $c_{i,\mathrm{a}}^{(2)}, i = 2, 3, L$
  (simple computer algebra)



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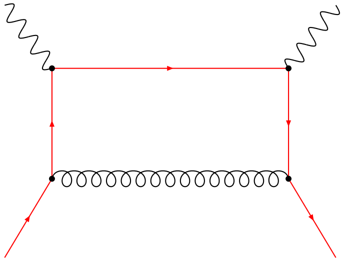

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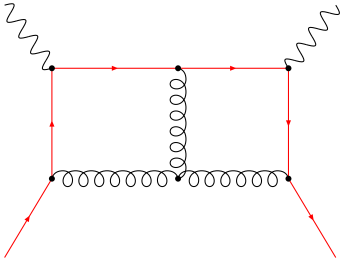
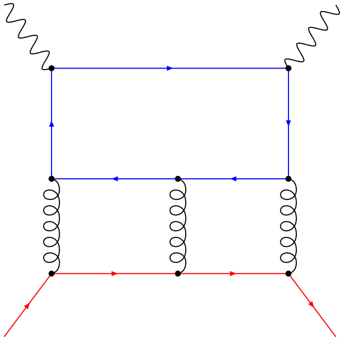


- **Three-loop** Feynman diagrams
  $\longrightarrow$ in total 9607 for $c_{i,\mathrm{a}}^{(3)}, i = 2, 3, L$
  (cutting edge technology $\longrightarrow$ computer algebra system FORM Vermaseren '89-'04)



Flavour classes (a subtlety)

- At three loops new contributions to coefficient functions emerge
- Charge factor distinguishes flavour classes $\langle e^k \rangle = \frac{1}{n_f} \sum_{i=1}^{n_f} e_i^k$

flavor factor	fl_2	fl_{11}	fl_{02}	fl_2^g	fl_{11}^g
non-singlet	1	$3\langle e \rangle$	0	–	–
singlet	1	$\frac{\langle e \rangle^2}{\langle e^2 \rangle}$	1	1	$\frac{\langle e \rangle^2}{\langle e^2 \rangle}$

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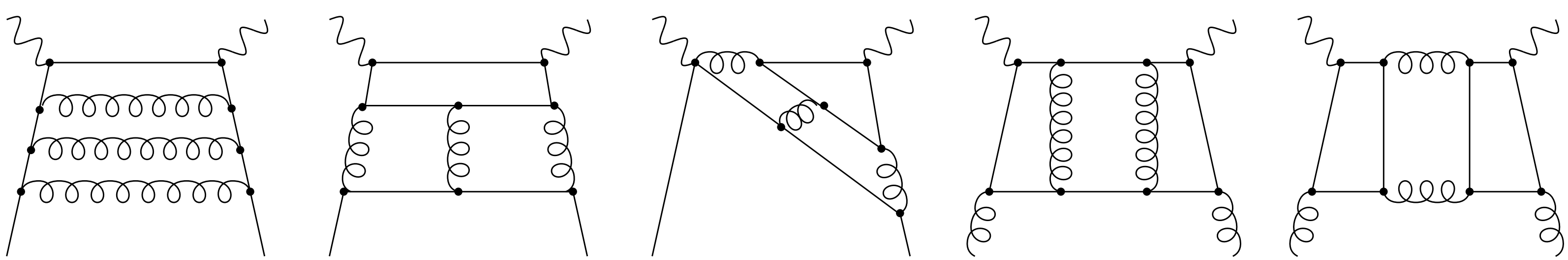
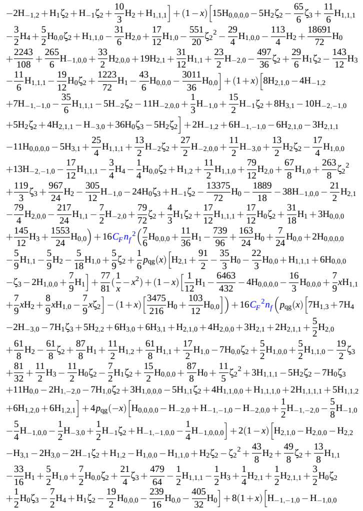
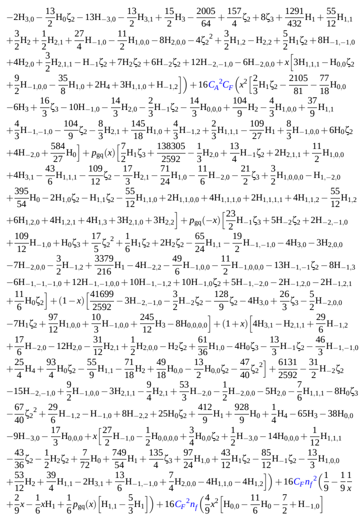
- Feynman diagrams (examples)



fl_2 $\quad fl_{02}$ $\quad fl_{11}$ $\quad fl_2^g$ $\quad fl_{11}^g$

Structure functions at third order





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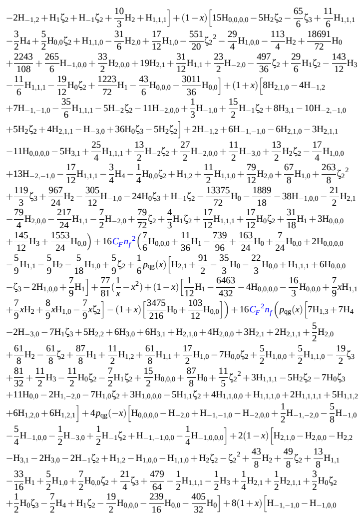
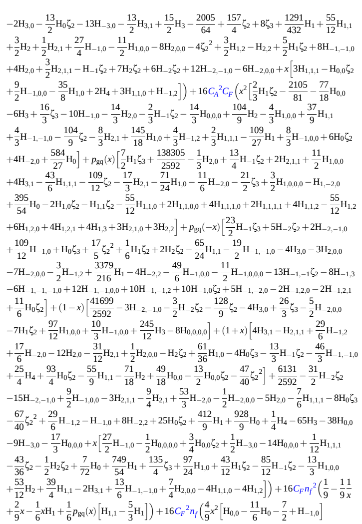




- Exact (analytical) results for coefficient functions of F_2 and F_L exceed $\mathcal{O}(100)$ pages
- Analytical results in x-space in terms of **harmonic polylogarithms** $H_{m_1,\dots,m_k}(x)$
  Goncharov '98; Borwein, Bradley,Broadhurst, Lisonek '99; Remiddi, Vermaseren '99
  - basic functions of lowest weight
    $H_0(x) = \ln x\,,\, H_1(x) = -\ln(1-x)\,,\, H_{-1}(x) = \ln(1+x)$
  - higher functions defined by recursion

$$H_{m_1,\dots,m_w}(x) = \int_0^x dz\; f_{m_1}(z)\; H_{m_2,\dots,m_w}(z)$$

$$f_0(x) = \frac{1}{x}\,, \qquad f_1(x) = \frac{1}{1-x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$

Easy-to-use parametrization (example)

- Combine exact limits for $x \to 0$ and $x \to 1$ with fit for intermediate x
  - fit quality better than one per mille
  - notation $L_0 = \ln(x)$, $L_1 = \ln(1-x)$, $x_1 = 1-x$

$$
\begin{aligned}
c^{(3)}_{L,\mathrm{ns}}(x) \;\cong\;& 512/27\, L_1^4 - 177.40\, L_1^3 + 650.6\, L_1^2 - 2729\, L_1 - 2220.5 - 7884\, x \\
& + 4168\, x^2 - (844.7\, L_0 + 517.3\, L_1) L_0 L_1 + (195.6\, L_1 - 125.3)\, x_1 L_1^3 \\
& + 208.3\, x L_0^3 - 1355.7\, L_0 - 7456/27\, L_0^2 - 1280/81\, L_0^3 + 0.113\, \delta(x_1) \\
+\; & n_f \,\{ 1024/81\, L_1^3 - 112.35\, L_1^2 + 344.1\, L_1 + 408.4 - 9.345\, x - 919.3\, x^2 \\
& + (239.7 + 20.63\, L_1)\, x_1 L_1^2 + (887.3 + 294.5\, L_0 - 59.14\, L_1) L_0 L_1 \\
& - 1792/81\, x L_0^3 + 200.73\, L_0 + 64/3\, L_0^2 + 0.006\, \delta(x_1) \} \\
+\; & n_f^2 \,\{ 3\, x L_1^2 + (6 - 25x) L_1 - 19 + (317/6 - 12\, \zeta_2)\, x - 6\, x L_0 L_1 + 6x\, \mathrm{Li}_2(x) \\
& + 9\, x L_0^2 - (6 - 50x) L_0 \}\, 64/81 \\
+\; & fl_{11}^{\mathrm{ns}}\, n_f \,\{ (107.0 + 321.05\, x - 54.62\, x^2)\, x_1 - 26.717 + 9.773\, L_0 \\
& + (363.8 + 68.32\, L_0) x L_0 - 320/81\, L_0^2 (2 + L_0) \}\, x
\end{aligned}
$$

The large x-limit: $x \to 1$

- Large x-limit for longitudinal coefficient functions
  - threshold resummation studied
    Akhoury, Sotiropoulos, Sterman '98; Akhoury, Sotiropoulos '03
  - no explicit prediction known for $c_L^{(3)}$
- Our conjecture

$$c_{L,\mathrm{ns}}^{(n)}(x)\Big|_{x\to 1} = c_{L,\mathrm{q}}^{(n)}(x)\Big|_{x\to 1} = \frac{2(2C_F)^n}{(n-1)!}\,\ln^{2n-2}(1-x) + \mathcal{O}(\ln^{2n-3}(1-x))$$

The small x-limit: $x \to 0$

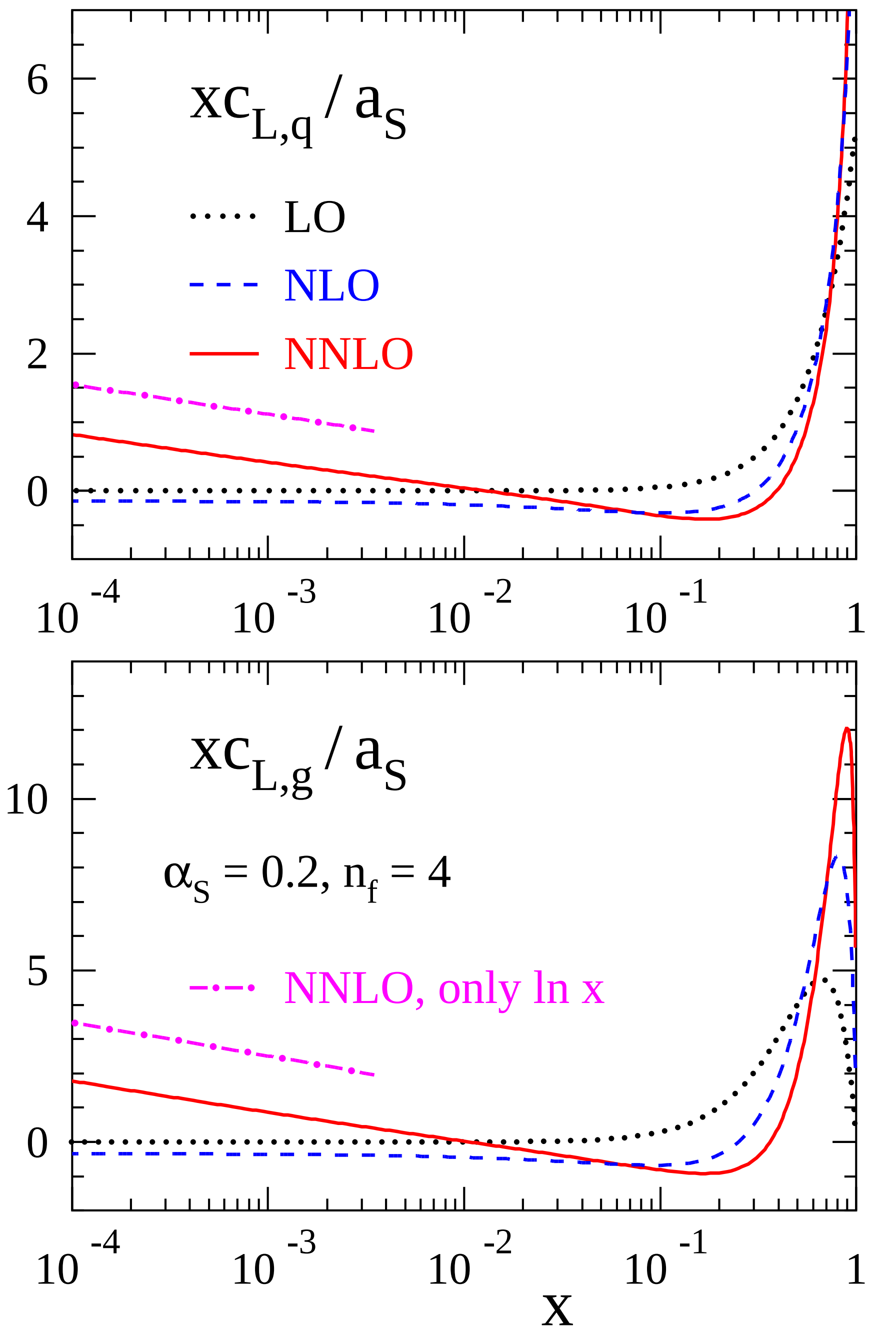
- Singlet coefficient functions
  - structure of $c^{(3)}_{L,\mathrm{ps}}$ and $c^{(3)}_{L,\mathrm{g}}$ at small x

$$c^{(3)}_{L,\mathrm{a},\to 0}(x) \;=\; E^{\mathrm{a}}_1 \,\frac{\ln x}{x} \;+\; E^{\mathrm{a}}_2 \,\frac{1}{x} \;+\; \mathcal{O}(\ln^4 x)$$

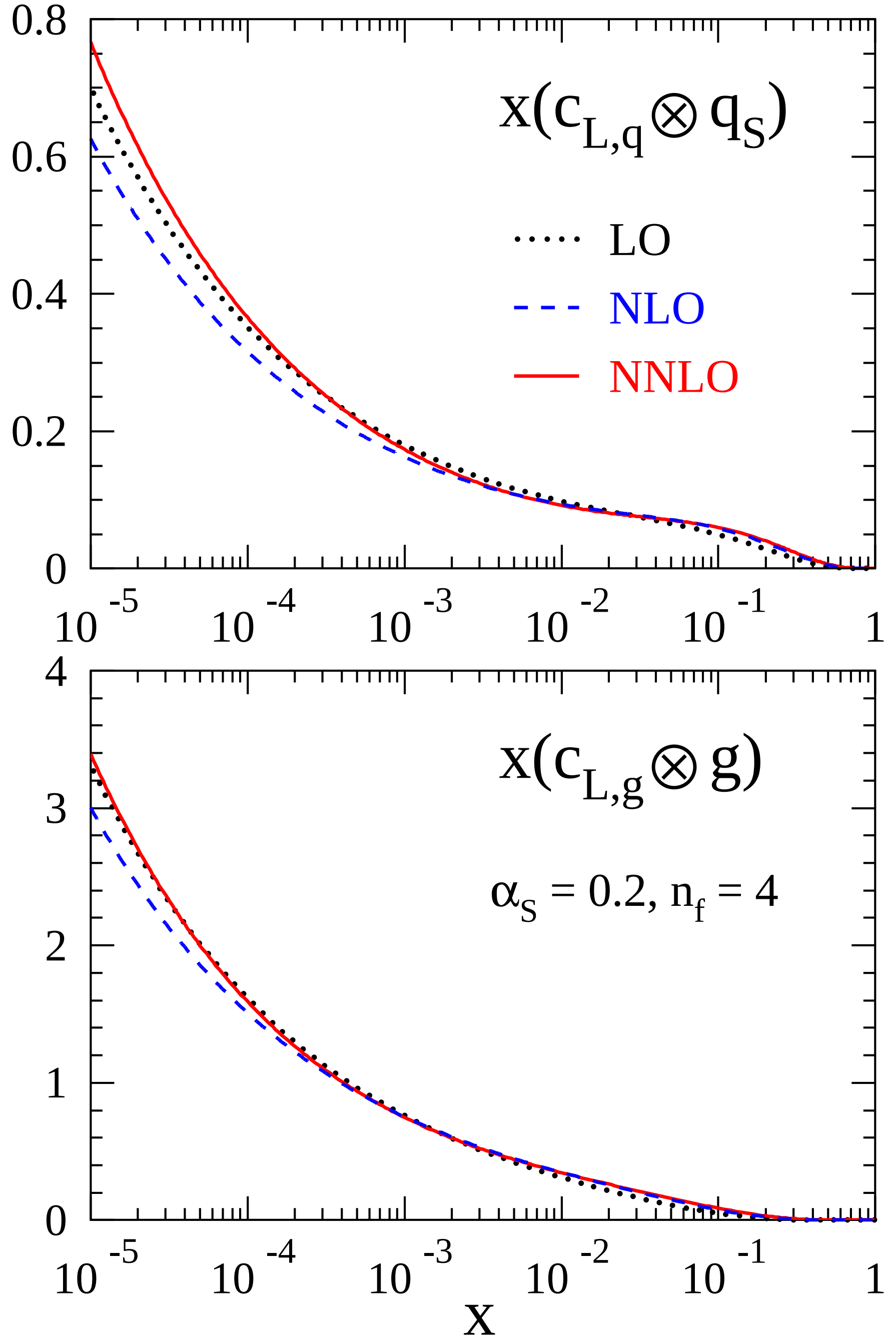
- coefficients E^{ps}_1, E^{g}_1 agree with prediction of small-x resummation
  Catani, Hautmann '94

$$E^{\mathrm{ps}}_1 \;\cong\; -182.00\, n_f \qquad\qquad E^{\mathrm{g}}_1 \;\cong\; -409.506\, n_f$$

$$E^{\mathrm{ps}}_2 \;\cong\; -885.53\, n_f + 40.239\, n_f^2 \qquad\qquad E^{\mathrm{g}}_2 \;\cong\; -2044.70\, n_f + 88.5037\, n_f^2$$



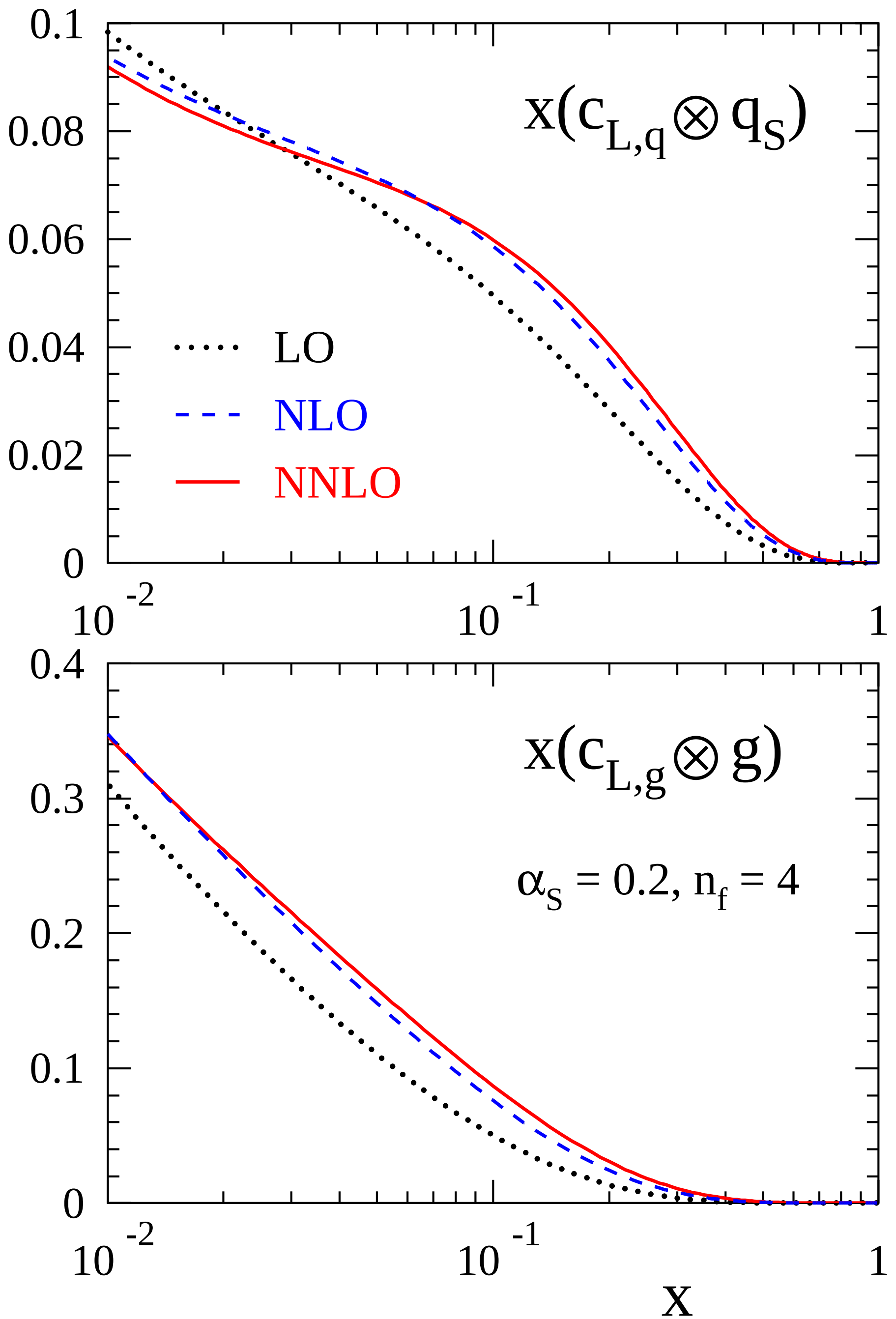
- Perturbative expansion of singlet-quark and gluon coefficient functions $c_{L,\mathrm{q}}$ and $c_{L,\mathrm{g}}$ for F_L with $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$ (results divided by $a_\mathrm{s} = \alpha_\mathrm{s}/(4\pi)$)
- LO and NLO contributions remarkably small
- At small-x $c_{L,\mathrm{q}}$ and $c_{L,\mathrm{g}}$ dominated by NNLO term
- Leading small-x term at NNLO results $xc^{(3)}_{L,a} \sim \ln x$



- Perturbative expansion of singlet-quark and gluon contributions to F_L for $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$ (results divided by $\langle e^2 \rangle$)
- Parametrization of singlet distributions (order independent)

$$xq_{\rm s}(x, \mu_0^2) = 0.6\, x^{-0.3}(1-x)^{3.5}\,(1 + 5.0\, x^{0.8})$$

$$xg(x, \mu_0^2) = 1.6\, x^{-0.3}(1-x)^{4.5}\,(1 - 0.6\, x^{0.3})$$



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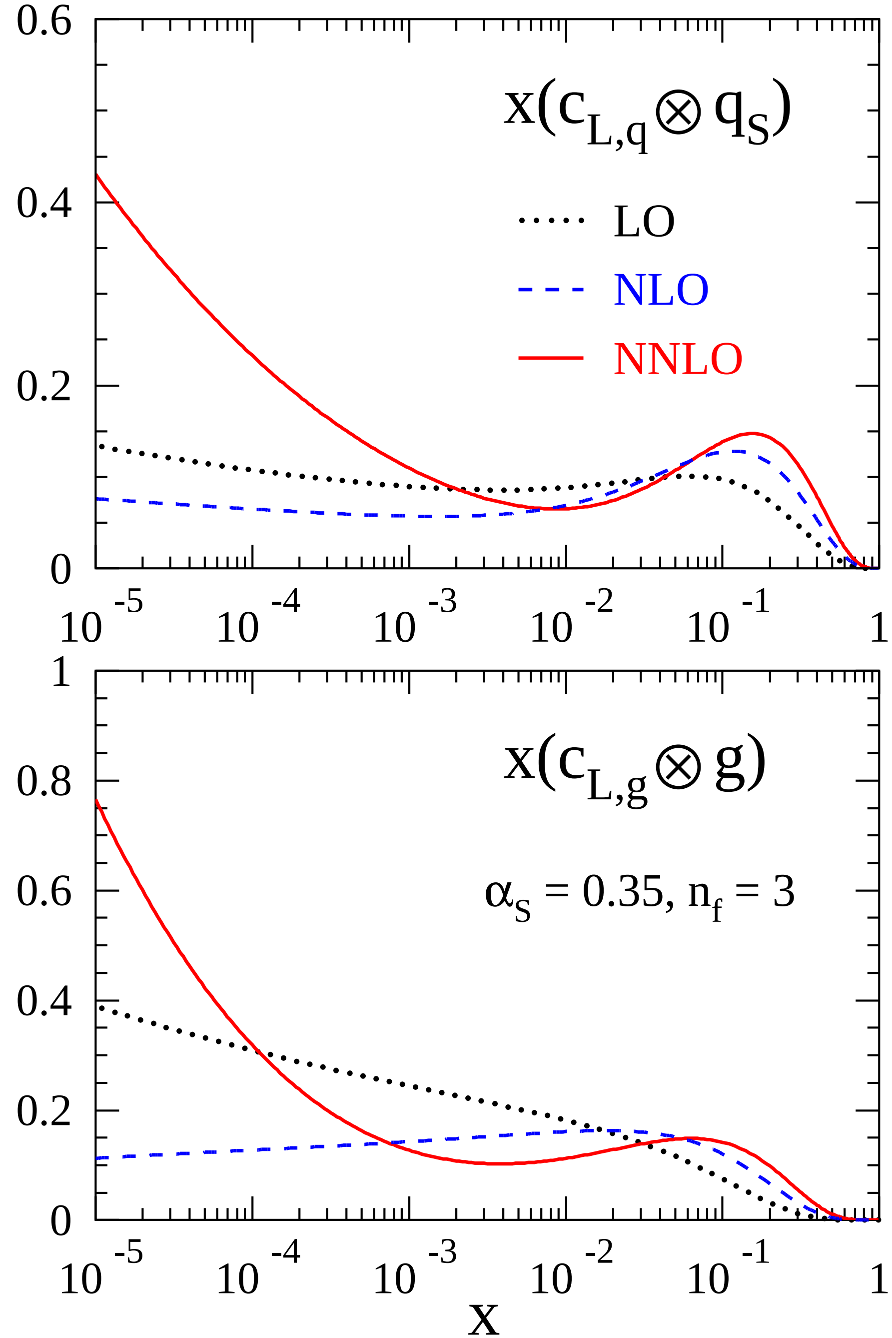
Zooming in at large x



- Perturbative expansion of singlet-quark and gluon contributions to F_L for $n_f = 3$ and $\alpha_s(\mu^2) = 0.35$ (results divided by $\langle e^2 \rangle$)
- Parametrization choice (order independent) similar to CTEQ5M and LesHouches benchmark Vogt, Salam '02

$$xq_s(x, \mu_0^2) =$$
$$0.6\, x^{-0.1} (1-x)^3 \, (1 + 10\, x^{0.8})$$
$$xg(x, \mu_0^2) =$$
$$1.2\, x^{-0.1} (1-x)^4 \, (1 + 1.5\, x)$$

Sensitivity at low scales $Q^2 \simeq 2$ GeV^2

Summary

- Complete QCD results at NNLO for deep-inelastic scattering
- Coefficient functions for F_L at third order
  - first complete three-loop calculation for single-scale hard scattering process
- Precision predictions ready for measurement of F_L at HERA
- Important constraint on gluon distribution through F_L