

The Longitudinal Structure Function at Third Order

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in collaboration with **J.A.M. Vermaseren** and **A. Vogt**, [hep-ph/0411112](#)

– Workshop HERA and the LHC, CERN, Geneva, Jan 20, 2005 –

Plan

- Motivation
- Method and Results
- Summary

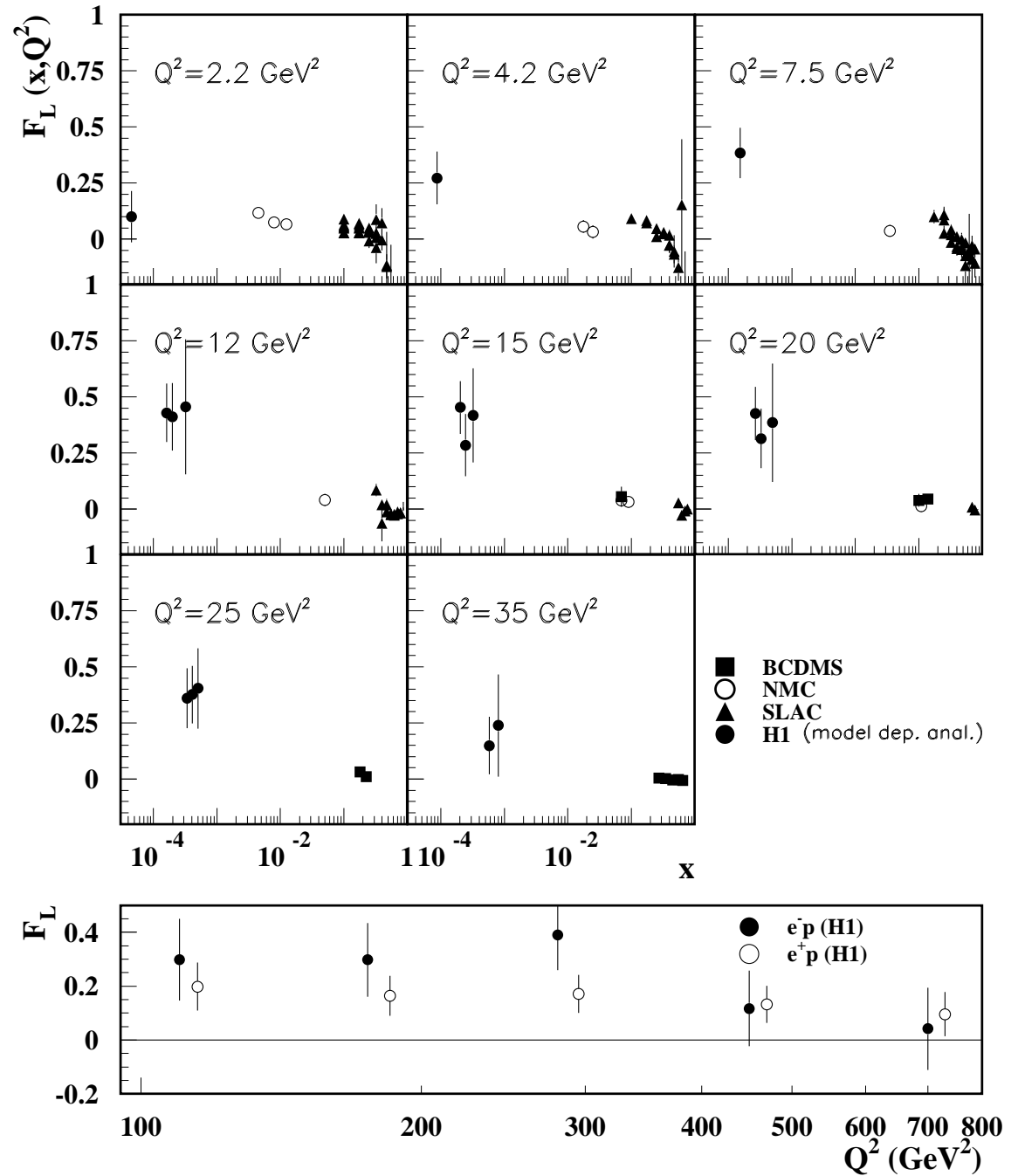
Motivation

Some facts about F_L

- Theory
 - Longitudinal structure function $F_L = F_2 - 2xF_1$ vanishes at tree level
 - Complete NNLO analysis of F_L (or $R = \frac{\sigma_T}{\sigma_L}$) requires three-loop coefficient functions
- Experiment
 - Various methods for measurement of F_L
 - R -ratio
 - derivative $\frac{\partial F_L}{\partial \ln y}$ (inelasticity y)
 - fits of the shape of F_L in cross section
 - HERA results for F_L
H1 collaboration '01; H1 collaboration '03; Klein '03; Lobodzinska '03

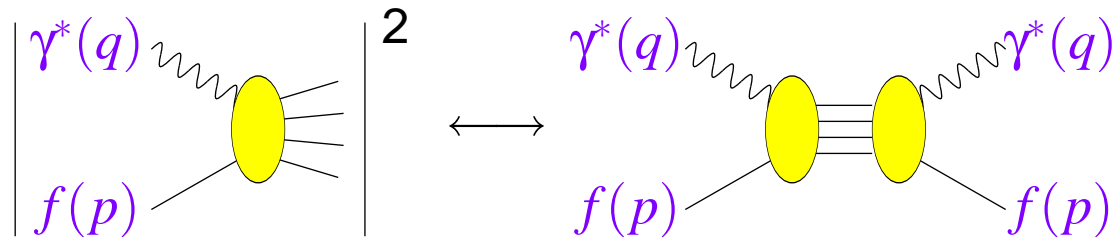
Data

- World data for F_L
- PDG 2004



Method and Results

Optical theorem



- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + i\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x, Q^2)$$

- Longitudinal structure function F_L

$$x^{-1} F_L = C_{L,ns} \otimes q_{ns} + \langle e^2 \rangle (C_{L,q} \otimes q_s + C_{L,g} \otimes g)$$

- number distributions of quarks and gluons q_i, g

The calculation (in a nut shell)

- Calculate coefficient functions $c_{2,a}$ and $c_{L,a}$ (Mellin moments) from Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

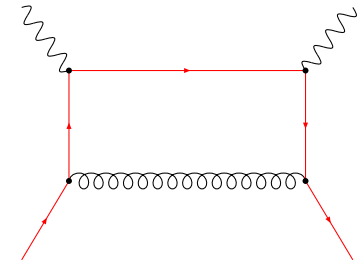
$$c_{i,a}^{(n)}(N) = \int_0^1 dx x^{N-1} c_{i,a}^{(n)}(x), \quad i = 2, 3, L$$

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- **One-loop** Feynman diagrams
→ in total 18 for $c_{i,a}^{(1)}, i = 2, 3, L$
(pencil + paper)



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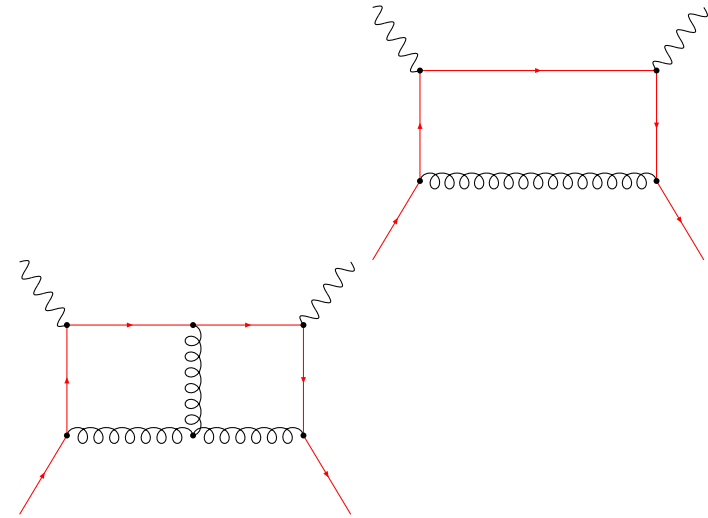
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→ in total 18 for $c_{i,a}^{(1)}, i = 2, 3, L$
(pencil + paper)

- **Two-loop** Feynman diagrams

→ in total 350 for $c_{i,a}^{(2)}, i = 2, 3, L$
(simple computer algebra)



The calculation (in a nut shell)

- Calculate coefficient functions $c_{2,a}$ and $c_{L,a}$ (Mellin moments) from Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

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- **One-loop** Feynman diagrams

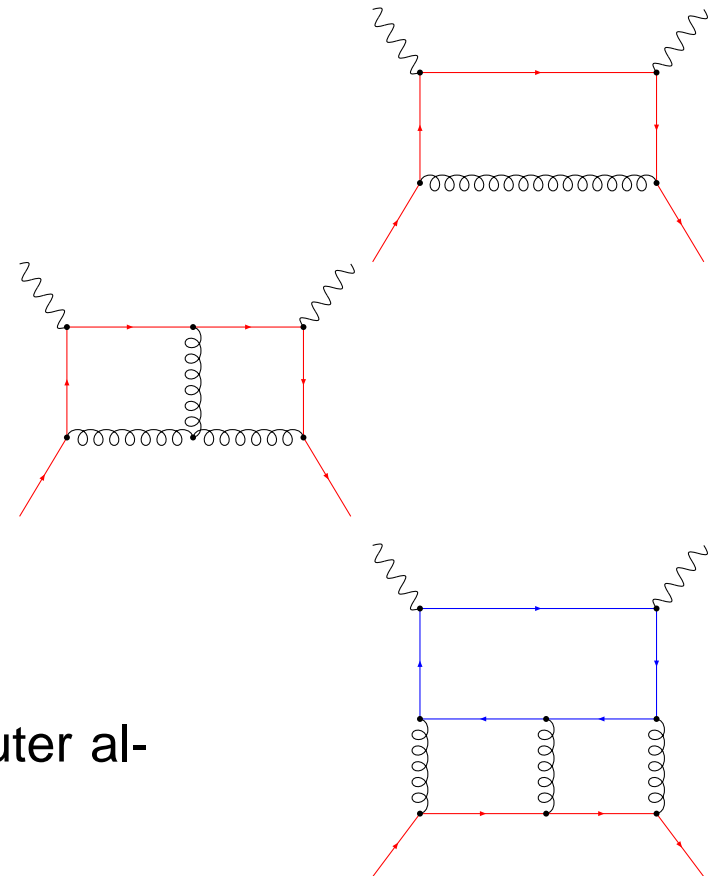
→ in total 18 for $c_{i,a}^{(1)}, i = 2, 3, L$
(pencil + paper)

- **Two-loop** Feynman diagrams

→ in total 350 for $c_{i,a}^{(2)}, i = 2, 3, L$
(simple computer algebra)

- **Three-loop** Feynman diagrams

→ in total 9607 for $c_{i,a}^{(3)}, i = 2, 3, L$
(cutting edge technology → computer algebra system FORM [Vermaseren '89-'04](#))



Flavour classes (a subtlety)

- At three loops new contributions to coefficient functions emerge

- Charge factor distinguishes flavour classes $\langle e^k \rangle = \frac{1}{n_f} \sum_{i=1}^{n_f} e_i^k$

flavor factor	fl_2	fl_{11}	fl_{02}	fl_2^g	fl_{11}^g
non-singlet	1	$3\langle e \rangle$	0	–	–
singlet	1	$\frac{\langle e \rangle^2}{\langle e^2 \rangle}$	1	1	$\frac{\langle e \rangle^2}{\langle e^2 \rangle}$

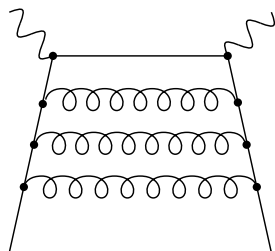
Flavour classes (a subtlety)

- At three loops new contributions to coefficient functions emerge

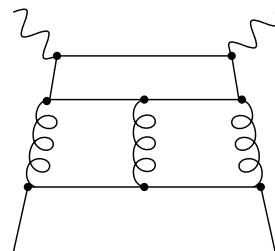
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flavor factor	fl_2	fl_{11}	fl_{02}	fl_2^g	fl_{11}^g
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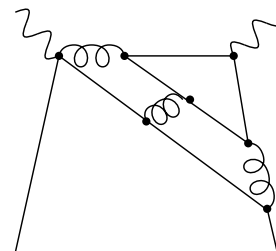
- Feynman diagrams (examples)



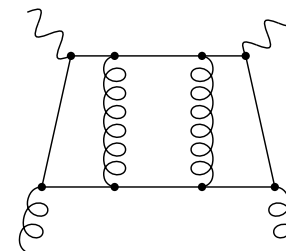
fl_2



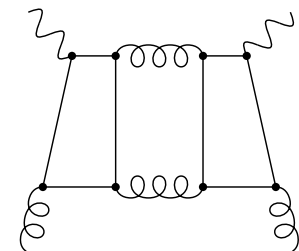
fl_{02}



fl_{11}



fl_2^g



fl_{11}^g

Structure functions at third order

$$\begin{aligned}
 & -2H_{-1,2} + H_1\zeta_5 + H_{-1}\zeta_4 + \frac{10}{3}H_2 + H_{1,1} + (1-x) \left[15H_{0,0,0} - 5H_2\zeta_5 - \frac{65}{6}\zeta_5 + \frac{11}{6}H_{1,1} \right. \\
 & - \frac{5}{2}H_4 + \frac{1}{2}H_0\zeta_5 + H_{1,1,0} - \frac{2}{3}H_{2,0} + \frac{17}{12}H_{1,0} - \frac{55}{20}\zeta_7 - \frac{29}{4}H_{1,0,0} - \frac{113}{4}H_4 + \frac{1869}{72}H_0 \\
 & \left. - \frac{2243}{108} - \frac{265}{6}H_{-1,0,0} + \frac{23}{6}H_{2,0,0} + 19H_{2,1} + \frac{31}{12}H_{1,1} + \frac{23}{2}H_{-2,0} - \frac{497}{36}\zeta_2 + \frac{29}{6}H_1\zeta_5 - \frac{143}{12}H_4 \right. \\
 & \left. - \frac{11}{6}H_{1,1,1} - \frac{19}{12}H_0\zeta_5 + \frac{1223}{72}H_1 - \frac{43}{6}H_{0,0,0} - \frac{301}{36}H_{0,0} + (1+x) \left[8H_{2,1,0} - 4H_{-1,2} \right. \right. \\
 & \left. \left. + 7H_{-1,-1,0} - \frac{35}{6}H_{1,1,1} - 5H_{-2}\zeta_5 - 11H_{-2,0,0} + \frac{15}{3}H_{-1,0} + \frac{15}{2}H_{-1}\zeta_5 + 8H_{1,1} - 10H_{-2,-1,0} \right. \right. \\
 & \left. \left. + 5H_2\zeta_5 + 4H_{2,1,1} - H_{-3,0} + 36H_0\zeta_5 - 5H_2\zeta_5 \right] + 2H_{-1,2} + 6H_{-1,-1,0} - 6H_{2,1,0} - 3H_{2,1,1} \right. \\
 & \left. - 11H_{0,0,0} - 5H_{1,1} + \frac{25}{4}H_{1,1,1} + \frac{13}{2}H_{-2}\zeta_5 + \frac{27}{4}H_{-2,0,0} + \frac{11}{2}H_{-3,0} + \frac{13}{2}H_2\zeta_5 - \frac{17}{4}H_{1,0,0} \right. \\
 & \left. + 13H_{-2,-1,0} - \frac{17}{12}H_{1,1,1} - \frac{3}{4}H_{-1} - \frac{1}{2}H_0\zeta_5 + H_{1,2} + \frac{11}{2}H_{1,1,0} + \frac{79}{8}H_{1,0} + \frac{67}{8}H_{1,0} + \frac{263}{8}\zeta_2^2 \right. \\
 & \left. - \frac{119}{2}\zeta_3 + \frac{967}{24}H_1 - \frac{305}{12}H_{-1,0} - 24H_0\zeta_5 + H_{-1}\zeta_5 - \frac{13375}{72}H_0 - \frac{1889}{72} - 38H_{-1,0,0} - \frac{21}{2}H_{2,1} \right. \\
 & \left. - \frac{79}{4}H_{2,0,0} - \frac{217}{24}H_{1,1} - \frac{79}{2}H_{-2,0} + \frac{79}{2}\zeta_5 + \frac{17}{12}H_{1,1,1} + \frac{17}{12}H_{1,1,0} + \frac{31}{18}H_1 + 3H_{0,0,0} \right. \\
 & \left. - \frac{145}{12}H_1 + \frac{1553}{24}H_{0,0} \right) + 16C_F\gamma_f \left(\frac{1}{6}H_{0,0,0} + \frac{36}{5}H_1 - \frac{739}{96} - \frac{163}{24}H_0 + \frac{34}{3}H_{0,0} + 2H_{0,0,0} \right. \\
 & \left. - \frac{5}{9}H_{1,1} - \frac{5}{9}H_2 - \frac{5}{18}H_{1,0} + \frac{5}{9}\zeta_5 + \frac{2}{9}p_{0,0}(x) \right) \left[H_{1,1} + \frac{11}{2} - \frac{739}{96} - \frac{163}{24}H_0 + \frac{34}{3}H_{0,0} + 2H_{0,0,0} \right. \\
 & \left. - \zeta_5 - 2H_{0,0} + \frac{7}{9}H_1 \right] + \frac{77}{81} \frac{1}{(x-x^2)} + (1-x) \left[\frac{1}{12}H_1 - \frac{6463}{432} - 4H_{0,0,0} - \frac{16}{3}H_{0,0,0} + \frac{7}{9}4H_{1,1} \right. \\
 & \left. + \frac{7}{9}H_2 + \frac{8}{9}H_{1,0} - \frac{7}{9}\zeta_5 \right] - (1+x) \left[\frac{8}{316}H_0 + \frac{103}{18}H_{0,0} \right] + 16C_F\gamma_f \left(p_{0,0}(x) \right) \left[7H_{1,1} + 7H_1 \right. \\
 & \left. - 2H_{-3,0} - 7H_1\zeta_5 + 5H_{2,1} + 6H_{1,0} + 6H_1 + H_{2,1,0} + 4H_{2,0,0} + 3H_{1,1} + 2H_{2,1,1} + \frac{5}{2}H_{1,0} \right. \\
 & \left. - \frac{61}{8}H_2 - \frac{61}{8}\zeta_5 + \frac{87}{8}H_1 - \frac{11}{2}H_{1,2} + \frac{61}{8}H_{1,1} + \frac{17}{2}H_{1,0} - 7H_0\zeta_5 + \frac{5}{2}H_{0,0,0} + \frac{5}{2}H_{1,1,0} - \frac{19}{2}\zeta_3 \right. \\
 & \left. - \frac{81}{32} + \frac{11}{2}H_1 - \frac{11}{2}H_0\zeta_5 - \frac{7}{2}H_1\zeta_5 + \frac{15}{8}H_{0,0,0} + \frac{87}{8}H_0 + \frac{11}{8}\zeta_5^2 + 3H_{1,1} - 5H_2\zeta_5 - 7H_0\zeta_5 \right. \\
 & \left. + 11H_{0,0} - 2H_{1,-2,0} - 7H_1\zeta_5 + 3H_{1,0,0} - 5H_{1,1}\zeta_5 + 4H_{1,1,0,0} + H_{1,1,1,0} + 2H_{1,1,1,1} + 5H_{1,1,2} \right. \\
 & \left. + 6H_{1,2,0} + 6H_{1,2,1} \right] + 4p_{0,0}(-x) \left[H_{0,0,0} - H_{-2,0} + H_{-1,-1,0} - H_{-2,0,0} + \frac{1}{2}H_{-1,-2,0} - \frac{5}{8}H_{-1,0} \right. \\
 & \left. - \frac{5}{4}H_{-1,0,0} - \frac{1}{2}H_{-3,0} + \frac{1}{2}H_{-1}\zeta_5 + H_{-1,-1,0,0} - \frac{1}{4}H_{-1,0,0,0} \right] + 2(1-x) \left[H_{2,1,0} - H_{2,0,0} - H_{2,2} \right. \\
 & \left. - H_{3,1} - 2H_{1,0} - 2H_{-1}\zeta_5 + H_{1,2} - H_{1,0,0} - H_{1,1,0} + H_2\zeta_5 - \zeta_5^2 + \frac{43}{8}H_0 + \frac{49}{8}\zeta_5 - \frac{13}{8}H_{1,1} \right. \\
 & \left. - \frac{33}{16}H_1 + \frac{5}{2}H_{1,0} + \frac{7}{2}H_{0,0,0} + \frac{21}{4}\zeta_5 + \frac{479}{64} - \frac{1}{2}H_{1,1,1} - \frac{1}{2}H_1 + \frac{1}{2}H_{2,1,1} + \frac{1}{2}H_{1,1,1} + \frac{1}{2}H_0\zeta_5 \right. \\
 & \left. + \frac{1}{2}H_0\zeta_5 - \frac{1}{2}H_1 + H_1\zeta_5 - \frac{19}{2}H_{0,0,0} - \frac{405}{16}H_{0,0} - \frac{405}{32}H_0 \right] + 8(1+x) \left[H_{-1,-1,0} - H_{-1,0,0} \right.
 \end{aligned}$$

Exact (analytical) results for coefficient functions of F_2 and F_L exceed $\mathcal{O}(100)$ pages

$$\begin{aligned}
 & -2H_{1,0} - \frac{13}{2}H_0\zeta_5 - 13H_{-3,0} - \frac{13}{2}H_{1,1} + \frac{15}{2}H_3 - \frac{2005}{64} + \frac{157}{4}\zeta_5 + 8\zeta_5 + \frac{1291}{432}H_1 + \frac{55}{12}H_{1,1} \\
 & + \frac{3}{2}H_2 + \frac{1}{2}H_{2,1} + \frac{27}{4}H_{-1,0} - \frac{11}{2}H_{0,0,0} - 8H_{2,0,0} - 4\zeta_5^2 + \frac{3}{2}H_{1,2} - H_{2,2} + \frac{5}{2}H_1\zeta_5 + 8H_{-1,-1,0} \\
 & + 4H_{2,0} + \frac{3}{2}H_{2,1,1} - H_{-1}\zeta_5 + 7H_2\zeta_5 + 6H_{-2}\zeta_5 + 12H_{-2,-1,0} - 6H_{-2,0,0} + x \left[3H_{1,1,1} - H_{0,0}\zeta_5 \right. \\
 & \left. + \frac{9}{8}H_{-1,0,0} - \frac{35}{8}H_{1,0} + 2H_4 + 3H_{1,1,0} + H_{-1,2} \right] + 16C_F^2 C_F \left(x^2 \left[\frac{5}{2}H_2\zeta_5 - \frac{2105}{81} - \frac{77}{18}H_0 \right. \right. \\
 & \left. \left. - 6H_1 + \frac{16}{3}\zeta_5 - 10H_{-1,0} - \frac{14}{3}H_{2,0} - 2H_{-1}\zeta_5 - \frac{14}{3}H_{0,0,0} + \frac{104}{9}H_2 - \frac{4}{3}H_{1,0,0} + \frac{37}{9}H_{1,1} \right. \right. \\
 & \left. \left. + \frac{4}{3}H_{-1,-1,0} - \frac{104}{9}\zeta_5 - \frac{5}{2}H_{2,1} + \frac{145}{18}H_{1,0} + \frac{4}{3}H_{-1,2} + \frac{2}{3}H_{1,1,1} - \frac{109}{32}H_1 + \frac{8}{3}H_{-1,0,0} + 6H_0\zeta_5 \right. \right. \\
 & \left. \left. + 4H_{-2,0} + \frac{384}{27}H_0 \right) + p_{0,0}(x) \left[\frac{2}{3}H_1\zeta_5 + \frac{138305}{2592} - \frac{1}{6}H_{2,0} + \frac{13}{12}H_{-1}\zeta_5 + 2H_{2,1,1} + \frac{11}{2}H_{1,0,0} \right. \right. \\
 & \left. \left. + 4H_{1,1} + \frac{43}{6}H_{1,1,1} - \frac{109}{12}\zeta_5 - \frac{17}{3}H_{2,1} - \frac{71}{24}H_{1,0} - \frac{11}{6}H_{-2,0} - \frac{21}{2}\zeta_5 + \frac{3}{2}H_{0,0,0} - H_{-1,2,0} \right. \right. \\
 & \left. \left. - \frac{395}{54}H_0 - 2H_1\zeta_5 - H_{1,1}\zeta_5 - \frac{55}{12}H_{1,0} + 2H_{1,1,0,0} + 4H_{1,1,1,0} + 2H_{1,1,1,1} + 4H_{1,1,2} - \frac{55}{12}H_{1,2} \right. \right. \\
 & \left. \left. + 6H_{1,2,0} + 4H_{1,2,1} + 4H_{1,3} + 3H_{2,1,0} + 3H_{2,2} \right] + p_{0,0}(-x) \left[\frac{23}{3}H_{-1}\zeta_5 + 5H_{-2}\zeta_5 + 2H_{-2,-1,0} \right. \right. \\
 & \left. \left. - \frac{102}{13}H_{-1,0} + H_0\zeta_5 + \frac{17}{3}\zeta_5^2 + \frac{1}{6}H_1\zeta_5 + 2H_2\zeta_5 - \frac{29}{24}H_{1,1} - \frac{19}{2}H_{-1,-1,0} - 4H_{1,0} - 3H_{2,0,0} \right. \right. \\
 & \left. \left. - 7H_{-2,0,0} - \frac{3}{2}H_{-1,2} - \frac{3379}{216} - \frac{1}{6}H_{-2,2} - \frac{49}{6}H_{-1,0,0} - \frac{11}{2}H_{-1,0,0,0} - 13H_{-1,-1}\zeta_5 - 8H_{-1,-1,3} \right. \right. \\
 & \left. \left. - 6H_{-1,-1,0} + 12H_{-1,-1,0,0} + 10H_{-1,-1,2} + 10H_{-1,0,0}\zeta_5 + \frac{5}{2}H_{-1,-2,0} - 2H_{-1,2,0} - 2H_{-1,2,1} \right. \right. \\
 & \left. \left. - \frac{11}{6}H_0\zeta_5 \right] + (1-x) \left[\frac{41699}{2592} - 3H_{-2,-1,0} - \frac{5}{2}H_{-2}\zeta_5 - \frac{128}{9}\zeta_5 - 4H_{1,0} + \frac{26}{3}\zeta_5 - \frac{5}{2}H_{-2,0,0} \right. \right. \\
 & \left. \left. - 7H_1\zeta_5 + \frac{92}{12}H_{1,0,0} + \frac{10}{3}H_{1,0,0} + \frac{245}{12}H_1 - 8H_{0,0,0} \right] + (1+x) \left[4H_{1,1} - H_{2,1,1} + \frac{29}{6}H_{-1,2} \right. \right. \\
 & \left. \left. - \frac{17}{6}H_{-2,0} - 12H_{2,0} - \frac{31}{12}H_{2,1} + \frac{1}{2}H_{2,0,0} - H_2\zeta_5 + \frac{61}{36}H_{1,0} - 4H_0\zeta_5 - \frac{13}{3}H_{-1}\zeta_5 + \frac{46}{3}H_{-1,-1,0} \right. \right. \\
 & \left. \left. + \frac{25}{4}H_1 + \frac{9}{2}H_0\zeta_5 - \frac{6131}{18}H_1 - \frac{71}{18}H_2 - \frac{49}{18}H_{0,0} - \frac{49}{18}H_{0,0}\zeta_5 - \frac{6131}{18}H_{-1,2} \right. \right. \\
 & \left. \left. - 15H_{-2,-1,0} + \frac{2}{3}H_{-1,0,0} - 3H_{2,1,1} - \frac{9}{2}H_{2,1} + \frac{23}{3}H_{-2,0} - \frac{1}{3}H_{-2,0,0} - 5H_{2,0} - \frac{2}{6}H_{1,1,1} - 8H_0\zeta_5 \right. \right. \\
 & \left. \left. - \frac{67}{20}\zeta_5^2 + \frac{29}{6}H_{-1,2} - H_{-1,0} + 8H_{-2,2} + 25H_0\zeta_5 + \frac{412}{9}H_1 + \frac{928}{9}H_0 + \frac{1}{4}H_4 - 65H_1 - 38H_{0,0} \right. \right. \\
 & \left. \left. - 9H_{-3,0} - \frac{13}{2}H_{0,0,0} + x \left[\frac{27}{2}H_{-1,0} - \frac{1}{2}H_{0,0,0,0} + \frac{3}{4}H_{0,0,0} + \frac{1}{2}H_{-3,0} - 14H_{0,0,0} + \frac{1}{4}H_{1,1} \right. \right. \right. \\
 & \left. \left. - \frac{43}{36}\zeta_5 + \frac{1}{2}H_2\zeta_5 + \frac{7}{2}H_0 + \frac{749}{36}H_1 + \frac{135}{4}\zeta_5 + \frac{97}{24}H_{1,0} + \frac{43}{12}H_{1,1} - \frac{85}{12}H_{-1}\zeta_5 - \frac{85}{12}H_{1,0} \right. \right. \\
 & \left. \left. - \frac{53}{12}H_2 + \frac{29}{4}H_{1,1} - 2H_{1,1} + \frac{13}{4}H_{-1,-1,0} + \frac{1}{4}H_{2,0,0} - 4H_{1,1,0} - 4H_{1,2} \right] + 16C_F\gamma_f \left(\frac{1}{9} - \frac{1}{9x} \right. \right. \\
 & \left. \left. - \frac{2}{9}x - \frac{1}{6}H_1 + \frac{1}{6}p_{0,0}(x) \left[H_{1,1} - \frac{5}{3}H_1 \right] \right) + 16C_F^2\gamma_f \left(\frac{d}{dx} \left[H_{0,0} - \frac{11}{6}H_0 - \frac{7}{2}H_{-1,0} \right] \right.
 \end{aligned}$$

Structure functions at third order

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 & -2H_{-1,2} + H_0\zeta_3 + H_{-1}\zeta_2 + \frac{10}{3}H_2 + H_{1,1} + (1-x) \left[15H_{0,0,0} - 5H_0\zeta_3 - \frac{65}{6}\zeta_3 + \frac{11}{6}H_{1,1} \right. \\
 & - \frac{5}{2}H_4 + \frac{1}{2}H_0\zeta_2 + H_{1,1,0} - \frac{2}{3}H_{2,0} + \frac{17}{12}H_{1,0} - \frac{55}{20}\zeta_2^2 - \frac{21}{4}H_{1,0,0} - \frac{113}{4}H_4 + \frac{1869}{72}H_0 \\
 & \frac{2243}{108} - \frac{265}{6}H_{-1,0,0} + \frac{23}{6}H_{2,0,0} + 19H_{2,1} + \frac{31}{12}H_{1,1} + \frac{23}{2}H_{-2,0} - \frac{497}{36}\zeta_2 + \frac{29}{6}H_0\zeta_2 - \frac{143}{12}H_4 \\
 & \frac{15}{6}H_{1,1,1} - \frac{19}{12}H_0\zeta_2 + \frac{1223}{72}H_1 - \frac{43}{6}H_{0,0,0} - \frac{3011}{36}H_{0,0} + (1+x) \left[8H_{2,1,0} - 4H_{-1,2} \right. \\
 & \left. + 7H_{-1,1,0} - \frac{35}{6}H_{1,1,1} - 5H_{-2}\zeta_2 - 11H_{-2,0,0} + \frac{1}{2}H_{-1,0} + \frac{15}{2}H_{-1}\zeta_2 + 8H_{1,1} - 10H_{-2,-1,0} \right. \\
 & \left. + 5H_0\zeta_2 + 4H_{2,1,1} - H_{-3,0} + 36H_0\zeta_3 - 5H_0\zeta_2^2 + 2H_{-1,2} + 6H_{-1,-1,0} - 6H_{2,1,0} - 3H_{2,1,1} \right. \\
 & \left. - 11H_{0,0,0} - 5H_{1,1} + \frac{25}{4}H_{1,1,1} + \frac{13}{2}H_{-2}\zeta_2 + \frac{27}{4}H_{-2,0,0} + \frac{11}{2}H_{-3,0} + \frac{13}{2}H_0\zeta_2 - \frac{17}{4}H_{1,0} \right. \\
 & \left. + 13H_{-2,-1,0} - \frac{17}{12}H_{1,1,1} - \frac{3}{4}H_4 - \frac{1}{2}H_0\zeta_2 + H_{1,2} + \frac{11}{2}H_{1,0,0} + \frac{79}{12}H_{1,0} + \frac{67}{8}H_{1,0,0} + \frac{263}{8}\zeta_2^2 \right. \\
 & \left. - \frac{119}{3}\zeta_3 + \frac{967}{24}H_2 - \frac{305}{12}H_{-1,0} - 24H_0\zeta_2 + H_{1,2} + \frac{17375}{72}H_0 - \frac{1889}{18} - 38H_{-1,0,0} - \frac{21}{2}H_{2,1} \right. \\
 & \left. - \frac{79}{4}H_{2,0,0} - \frac{217}{24}H_{1,1} - \frac{79}{2}H_{-2,0} + \frac{79}{2}\zeta_2 + \frac{17}{12}H_{1,1,1} + \frac{17}{12}H_{1,0} - \frac{31}{18}H_4 + 3H_{0,0} \right. \\
 & \left. - \frac{145}{12}H_1 + \frac{1553}{24}H_0 \right] + 16C_F\gamma_f \left(\frac{7}{6}H_{0,0,0} + \frac{11}{36}H_1 - \frac{739}{96} + \frac{163}{24}H_0 + \frac{1}{24}H_{0,0} + 2H_{0,0,0} \right. \\
 & \left. - \frac{5}{2}H_{1,1} - \frac{5}{2}H_2 - \frac{5}{18}H_{1,0} + \frac{5}{2}\zeta_2 + \frac{1}{2}p_{0,0}(x) \right) \left[H_{1,1} + \frac{91}{2} - \frac{35}{3}H_0 - \frac{22}{3}H_{0,0} + H_{1,1,1} + 6H_{0,0} \right. \\
 & \left. - \zeta_3 - 2H_{0,0} + \frac{7}{2}H_1 \right] + \frac{77}{81}(1-x^2) + (1-x) \left[\frac{1}{12}H_1 - \frac{6463}{432} - 4H_{0,0,0} - \frac{16}{3}H_{0,0,0} + \frac{7}{9}H_{1,1} \right. \\
 & \left. + \frac{7}{6}H_2 + \frac{8}{9}H_{1,0} - \frac{7}{9}\zeta_2^2 \right] - (1+x) \left[\frac{3475}{216}H_0 + \frac{103}{18}H_{0,0} \right] + 16C_F^2\gamma_f \left(p_{0,0}(x) \right) \left[7H_{1,3} + 7H_4 \right. \\
 & \left. - 2H_{-3,0} - 7H_0\zeta_3 + 5H_{2,1} + 6H_{3,0} + 6H_{1,1} + H_{2,1,0} + 4H_{2,0,0} + 3H_{1,1} + 2H_{2,1,1} + \frac{5}{2}H_{2,0} \right. \\
 & \left. - \frac{61}{8}H_2 - \frac{61}{8}\zeta_2 - \frac{87}{8}H_1 - \frac{11}{2}H_{1,2} + \frac{61}{8}H_{1,1} + \frac{17}{8}H_{1,0} - 7H_0\zeta_2 + \frac{5}{2}H_{0,0,0} + \frac{5}{2}H_{1,0,0} - \frac{19}{8}\zeta_3 \right. \\
 & \left. - \frac{81}{32} + \frac{11}{24}H_1 - \frac{11}{24}H_0\zeta_2 - \frac{7}{24}H_0\zeta_3 + \frac{12}{24}H_{0,0,0} + \frac{87}{8}H_0 + \frac{11}{8}\zeta_2^2 + 3H_{1,1} - 5H_0\zeta_2 - 7H_0\zeta_3 \right. \\
 & \left. + 11H_{0,0} - 2H_{-2,0} - 7H_1\zeta_2 + 3H_{1,0,0} - 5H_{1,1}\zeta_2 + 4H_{1,0,0} + H_{1,1,0} + 2H_{1,1,1} + 5H_{1,1,2} \right. \\
 & \left. + 6H_{1,2,0} + 6H_{1,2,1} \right] + 4p_{0,0}(-x) \left[H_{0,0,0} - H_{-2,0} + H_{-1,-1,0} - H_{-2,0,0} + \frac{1}{2}H_{-1,-2,0} - \frac{5}{8}H_{-1,0} \right. \\
 & \left. - \frac{5}{2}H_{-1,0,0} - \frac{1}{2}H_{-3,0} + \frac{1}{2}H_{-2,0} + H_{-1,-1,0,0} - \frac{1}{2}H_{-1,0,0} \right] + 2(1-x) \left[H_{2,1,0} - H_{2,0,0} - H_{2,2} \right. \\
 & \left. - \frac{1}{2}H_{1,0} - 2H_{-1,0} - 2H_{-1}\zeta_2 + H_2 - H_{1,0,0} - H_{1,1,0} + H_0\zeta_2 - \zeta_2^2 + \frac{43}{8}H_0 + \frac{13}{8}\zeta_2 + \frac{13}{8}H_{1,1} \right. \\
 & \left. - \frac{33}{16}H_1 + \frac{5}{2}H_{1,0} + \frac{7}{2}H_0\zeta_2 + \frac{21}{4}\zeta_3 + \frac{479}{64}H_{1,1,1} - \frac{1}{2}H_4 + \frac{21}{4}H_{2,1} + \frac{1}{2}H_{2,1,1} + \frac{1}{2}H_0\zeta_2 \right. \\
 & \left. + \frac{1}{2}H_0\zeta_3 - \frac{1}{2}H_4 + H_0\zeta_2 - \frac{19}{2}H_{0,0} - \frac{405}{16}H_{0,0} \right] + 8(1+x) \left[H_{-1,-1,0} - H_{-1,0,0} \right.
 \end{aligned}$$

- Exact (analytical) results for coefficient functions of F_2 and F_L exceed $\mathcal{O}(100)$ pages

- Analytical results in x -space in terms of

harmonic polylogarithms $H_{m_1, \dots, m_k}(x)$

Goncharov '98; Borwein, Bradley, Broadhurst, Lisonek '99; Remiddi, Vermaseren '99

$$\begin{aligned}
 & -2H_{3,0} - \frac{13}{2}H_0\zeta_2 - 13H_{-3,0} - \frac{13}{2}H_{1,1} + \frac{15}{2}H_3 - \frac{2005}{64} + \frac{157}{4}\zeta_2 + 8\zeta_3 + \frac{1291}{432}H_1 + \frac{55}{12}H_{1,1} \\
 & + \frac{3}{2}H_2 + \frac{1}{2}H_{2,1} + \frac{27}{4}H_{-1,0} - \frac{11}{2}H_{0,0,0} - 8H_{2,0,0} - 4\zeta_2^2 + \frac{3}{2}H_{2,1,0} - H_{2,2} + \frac{5}{2}H_0\zeta_2 + 8H_{-1,-1,0} \\
 & + 4H_{2,0} + \frac{3}{2}H_{2,1,1} - H_{-1}\zeta_3 + 7H_0\zeta_2 + 6H_{-2}\zeta_2 + 12H_{-2,-1,0} - 6H_{-2,0,0} + x \left[3H_{1,1,1} - H_{0,0}\zeta_2 \right. \\
 & \left. + \frac{9}{8}H_{-1,0,0} - \frac{35}{8}H_{1,0} + 2H_4 + 3H_{1,1,0} + H_{-1,2} \right] + 16C_F^2C_F \left(x^2 \left[\frac{5}{2}H_0\zeta_2 - \frac{2105}{81} - \frac{77}{18}H_0 \right. \right. \\
 & \left. \left. - 6H_1 + \frac{16}{3}\zeta_2 - 10H_{-1,0} - \frac{14}{3}H_{2,0} - 2H_{-1}\zeta_2 - \frac{15}{3}H_{0,0,0} + \frac{104}{9}H_2 - \frac{4}{3}H_{1,0,0} + \frac{27}{8}H_1 \right. \right. \\
 & \left. \left. + \frac{4}{3}H_{-1,1,0} - \frac{104}{9}\zeta_2 - \frac{5}{2}H_{2,1} + \frac{145}{18}H_{1,0} + \frac{4}{3}H_{1,1,2} + \frac{1}{2}H_{1,1,1} - \frac{109}{27}H_1 + \frac{8}{3}H_{-1,0,0} + 6H_0\zeta_2 \right. \right. \\
 & \left. \left. + 4H_{-2,0} + \frac{384}{27}H_0 \right) + p_{0,0}(x) \left[\frac{2}{3}H_0\zeta_2 + \frac{138305}{2592} - \frac{1}{6}H_{0,0,0} + \frac{13}{4}H_{-1}\zeta_2 + 2H_{2,1,1} + \frac{11}{2}H_{1,0,0} \right. \right. \\
 & \left. \left. + 4H_{1,1} + \frac{43}{6}H_{1,1,1} - \frac{109}{12}\zeta_2 - \frac{17}{3}H_{2,1} - \frac{71}{24}H_{1,0} - \frac{11}{6}H_{-2,0} - \frac{21}{2}\zeta_3 + \frac{5}{3}H_{1,0,0,0} - H_{1,-2,0} \right. \right. \\
 & \left. \left. + \frac{392}{54}H_0 - 2H_1\zeta_2 - H_{1,1}\zeta_2 - \frac{55}{12}H_{1,0,0} + 2H_{1,1,0,0} + 4H_{1,1,1,0} + 2H_{1,1,1,1} + 4H_{1,1,2} - \frac{55}{12}H_{1,2} \right. \right. \\
 & \left. \left. + 6H_{1,2,0} + 4H_{1,2,1} + 4H_{1,3} + 3H_{2,1,0} + 3H_{2,2} \right) + p_{0,0}(-x) \left[\frac{23}{3}H_{-1}\zeta_2 + 5H_{-2}\zeta_2 + 2H_{-2,-1,0} \right. \right. \\
 & \left. \left. - \frac{102}{13}H_{-1,0} + H_0\zeta_2^2 + \frac{17}{3}\zeta_2^2 + \frac{1}{6}H_0\zeta_2 + 2H_0\zeta_3 - \frac{29}{24}H_{1,1} - \frac{19}{6}H_{-1,-1,0} - 4H_{1,0} - 3H_{2,0,0} \right. \right. \\
 & \left. \left. - 7H_{-2,0,0} - \frac{3}{2}H_{-1,2} - \frac{3379}{216}H_{-1,1} - 4H_{-2,2} - \frac{49}{6}H_{-1,0,0} - \frac{11}{2}H_{-1,0,0,0} - 13H_{-1,-1}\zeta_2 - 8H_{-1,3} \right. \right. \\
 & \left. \left. - 6H_{-1,-1,0} + 12H_{-1,-1,0,0} + 10H_{-1,-1,2} + 10H_{-1,0,0}\zeta_2 + 5H_{-1,2,0} - 2H_{-1,2,0} - 2H_{-1,2,1} \right. \right. \\
 & \left. \left. + \frac{11}{6}H_0\zeta_2^2 + (1-x) \left[\frac{41699}{2592} - 3H_{-2,-1,0} - \frac{3}{2}H_{-2}\zeta_2 - \frac{128}{9}\zeta_2 - 4H_{1,0} + \frac{26}{3}\zeta_3 - \frac{5}{2}H_{-2,0,0} \right. \right. \right. \\
 & \left. \left. - 7H_0\zeta_2 + \frac{92}{12}H_{0,0,0} + \frac{10}{3}H_{1,0,0} + \frac{242}{12}H_{0,0} - 8H_{0,0,0,0} \right) + (1+x) \left[4H_{1,1} - H_{2,1,1} + \frac{29}{6}H_{1,2} \right. \right. \\
 & \left. \left. - \frac{17}{6}H_{-2,0} - 12H_{2,0} - \frac{31}{12}H_{2,1} + 2H_{0,0,0} - H_2\zeta_2 + \frac{61}{36}H_{1,0} - 4H_0\zeta_2 - \frac{13}{3}H_{-1}\zeta_2 - \frac{46}{3}H_{-1,1,0} - \frac{25}{4} \right. \right. \\
 & \left. \left. + \frac{25}{4}H_4 + \frac{92}{3}H_0\zeta_2 - \frac{65}{6}H_{1,1} - \frac{71}{18}H_3 + \frac{49}{18}H_{0,0} - \frac{13}{18}H_{0,0,0} - \frac{49}{6}\zeta_2^2 \right] + 6131 - \frac{13}{9}H_{-2}\zeta_2 \right. \\
 & \left. - 15H_{-2,-1,0} + \frac{2}{3}H_{-2,0,0} - 3H_{2,1,1} - \frac{9}{2}H_{2,1} + \frac{23}{3}H_{-2,0} - \frac{1}{3}H_{-2,0,0} - 5H_{2,0} - \frac{5}{6}H_{1,1,1} - 8H_0\zeta_2 \right. \\
 & \left. - \frac{67}{20}\zeta_2^2 + \frac{29}{6}H_{-1,2} - H_{-1,0} + 8H_{-2,2} + 25H_0\zeta_2 + \frac{412}{9}H_1 + \frac{928}{9}H_0 + \frac{1}{9}H_4 - 65H_3 - 38H_{0,0} \right. \\
 & \left. - 9H_{-3,0} - \frac{13}{2}H_{0,0,0} + x \left[\frac{27}{2}H_{-1,0} - \frac{1}{2}H_{0,0,0,0} + \frac{3}{4}H_{0,0,0,0} + \frac{1}{2}H_{-3,0} - 14H_{0,0,0} + \frac{1}{2}H_{1,1} \right. \right. \\
 & \left. \left. - \frac{43}{36}\zeta_2 + \frac{1}{2}H_0\zeta_2 + \frac{7}{24}H_0 + \frac{749}{4}H_1 + \frac{135}{4} + \frac{97}{24}H_{1,0} + \frac{43}{12}H_{1,0,0} - \frac{85}{12}H_{-1}\zeta_2 - \frac{13}{6}H_{1,0,0} \right. \right. \\
 & \left. \left. + \frac{53}{12}H_2 + \frac{29}{4}H_{1,1} - 2H_{1,1} + \frac{13}{4}H_{-1,-1,0} + \frac{1}{4}H_{2,0,0} - 4H_{1,1,0} - 4H_{1,2} \right] + 16C_F^2\gamma_f \left(\frac{1}{9} - \frac{1}{9}x \right. \right. \\
 & \left. \left. + \frac{1}{2}x - \frac{1}{6}H_1 + \frac{1}{6}p_{0,0}(x) \left[H_{1,1} - \frac{5}{3}H_1 \right] \right) + 16C_F^2\gamma_f \left(\frac{4}{9}x^2 \left[H_{0,0} - \frac{11}{6}H_0 - \frac{7}{2} + H_{-1,0} \right] \right.
 \end{aligned}$$

- basic functions of lowest weight

$$H_0(x) = \ln x, H_1(x) = -\ln(1-x), H_{-1}(x) = \ln(1+x)$$

- higher functions defined by recursion

$$H_{m_1, \dots, m_w}(x) = \int_0^x dz f_{m_1}(z) H_{m_2, \dots, m_w}(z)$$

$$f_0(x) = \frac{1}{x}, f_1(x) = \frac{1}{1-x}, f_{-1}(x) = \frac{1}{1+x}$$

Easy-to-use parametrization (example)

- Combine exact limits for $x \rightarrow 0$ and $x \rightarrow 1$ with fit for intermediate x
 - fit quality better than one per mille
 - notation $L_0 = \ln(x)$, $L_1 = \ln(1 - x)$, $x_1 = 1 - x$

$$\begin{aligned}
 c_{L,\text{ns}}^{(3)}(x) \cong & 512/27 L_1^4 - 177.40 L_1^3 + 650.6 L_1^2 - 2729 L_1 - 2220.5 - 7884 x \\
 & + 4168 x^2 - (844.7 L_0 + 517.3 L_1)L_0L_1 + (195.6 L_1 - 125.3) x_1 L_1^3 \\
 & + 208.3 x L_0^3 - 1355.7 L_0 - 7456/27 L_0^2 - 1280/81 L_0^3 + 0.113 \delta(x_1) \\
 & + n_f \{ 1024/81 L_1^3 - 112.35 L_1^2 + 344.1 L_1 + 408.4 - 9.345 x - 919.3 x^2 \\
 & + (239.7 + 20.63 L_1) x_1 L_1^2 + (887.3 + 294.5 L_0 - 59.14 L_1)L_0L_1 \\
 & - 1792/81 x L_0^3 + 200.73 L_0 + 64/3 L_0^2 + 0.006 \delta(x_1) \} \\
 & + n_f^2 \{ 3 x L_1^2 + (6 - 25x)L_1 - 19 + (317/6 - 12 \zeta_2) x - 6 x L_0L_1 + 6x \text{Li}_2(x) \\
 & + 9 x L_0^2 - (6 - 50x)L_0 \} 64/81 \\
 & + fl_{11}^{\text{ns}} n_f \{ (107.0 + 321.05 x - 54.62 x^2) x_1 - 26.717 + 9.773 L_0 \\
 & + (363.8 + 68.32 L_0)xL_0 - 320/81 L_0^2(2 + L_0) \} x
 \end{aligned}$$

The large x -limit: $x \rightarrow 1$

- Large x -limit for longitudinal coefficient functions
 - threshold resummation studied
Akhoury, Sotiropoulos, Sterman '98; Akhoury, Sotiropoulos '03
 - no explicit prediction known for $c_L^{(3)}$
- Our conjecture

$$c_{L,\text{ns}}^{(n)}(x) \Big|_{x \rightarrow 1} = c_{L,\text{q}}^{(n)}(x) \Big|_{x \rightarrow 1} = \frac{2(2C_F)^n}{(n-1)!} \ln^{2n-2}(1-x) + \mathcal{O}(\ln^{2n-3}(1-x))$$

The small x -limit: $x \rightarrow 0$

- Singlet coefficient functions
 - structure of $c_{L,\text{ps}}^{(3)}$ and $c_{L,\text{g}}^{(3)}$ at small x

$$c_{L,\text{a},\rightarrow 0}^{(3)}(x) = E_1^{\text{a}} \frac{\ln x}{x} + E_2^{\text{a}} \frac{1}{x} + \mathcal{O}(\ln^4 x)$$

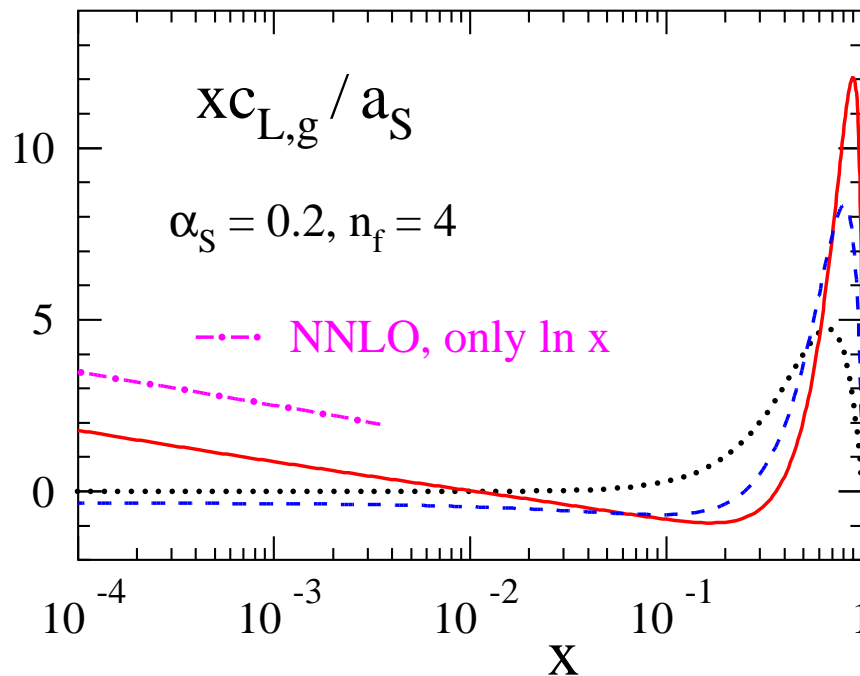
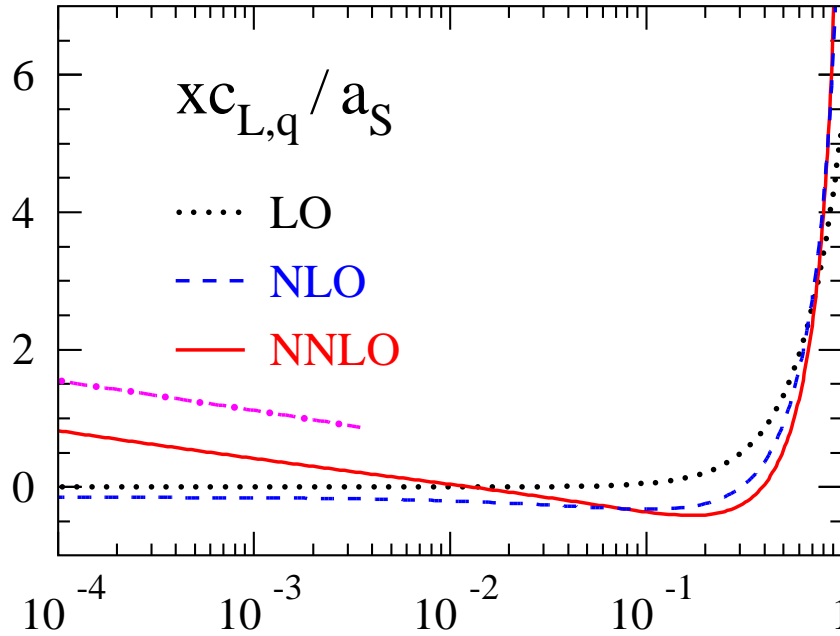
- coefficients E_1^{ps} , E_1^{g} agree with prediction of small- x resummation
Catani, Hautmann '94

$$E_1^{\text{ps}} \cong -182.00 n_f$$

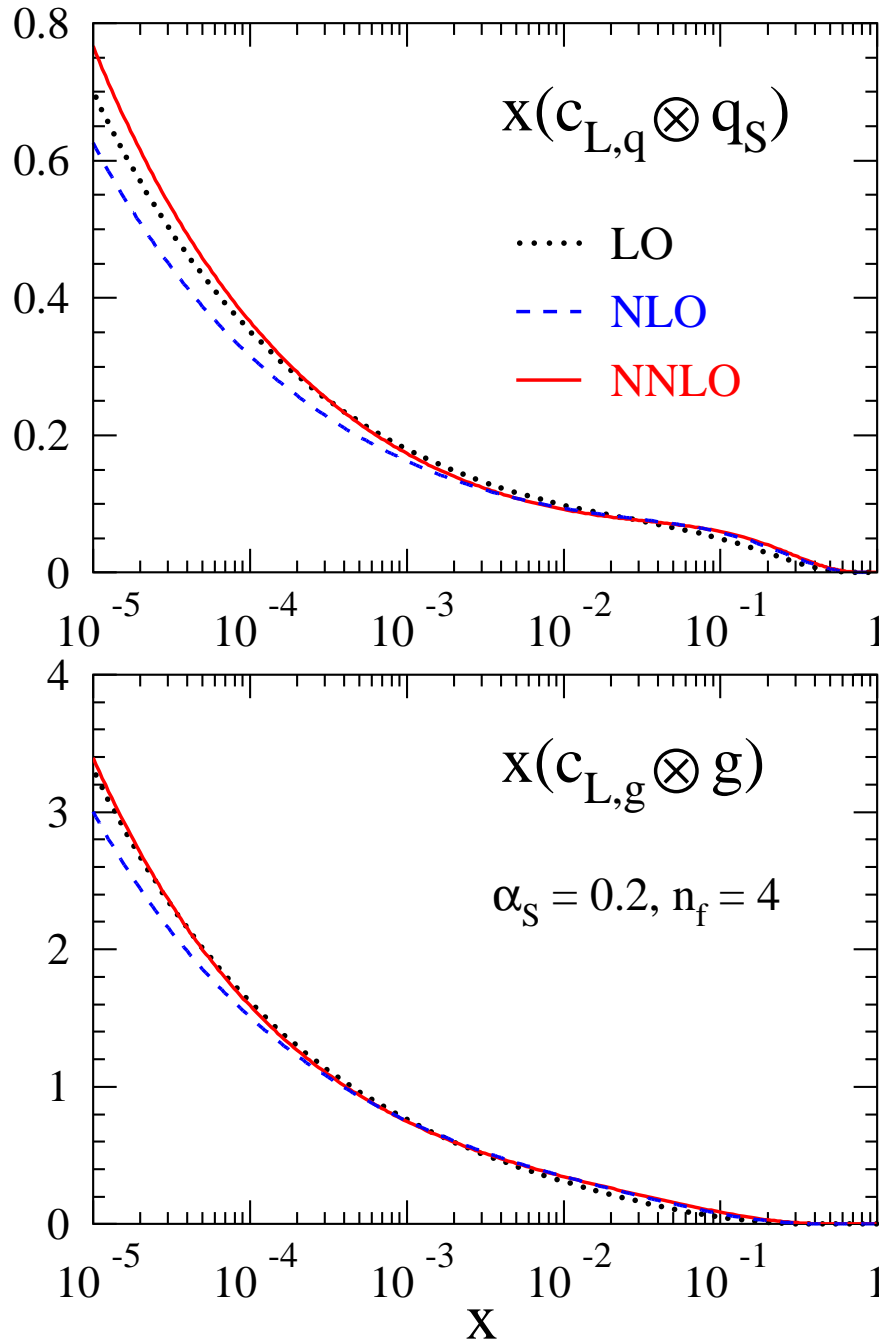
$$E_1^{\text{g}} \cong -409.506 n_f$$

$$E_2^{\text{ps}} \cong -885.53 n_f + 40.239 n_f^2$$

$$E_2^{\text{g}} \cong -2044.70 n_f + 88.5037 n_f^2$$



- Perturbative expansion of singlet-quark and gluon coefficient functions $c_{L,q}$ and $c_{L,g}$ for F_L with $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$ (results divided by $a_s = \alpha_s/(4\pi)$)
- LO and NLO contributions remarkably small
- At small- x $c_{L,q}$ and $c_{L,g}$ dominated by NNLO term
- Leading small- x term at NNLO results $x c_{L,a}^{(3)} \sim \ln x$



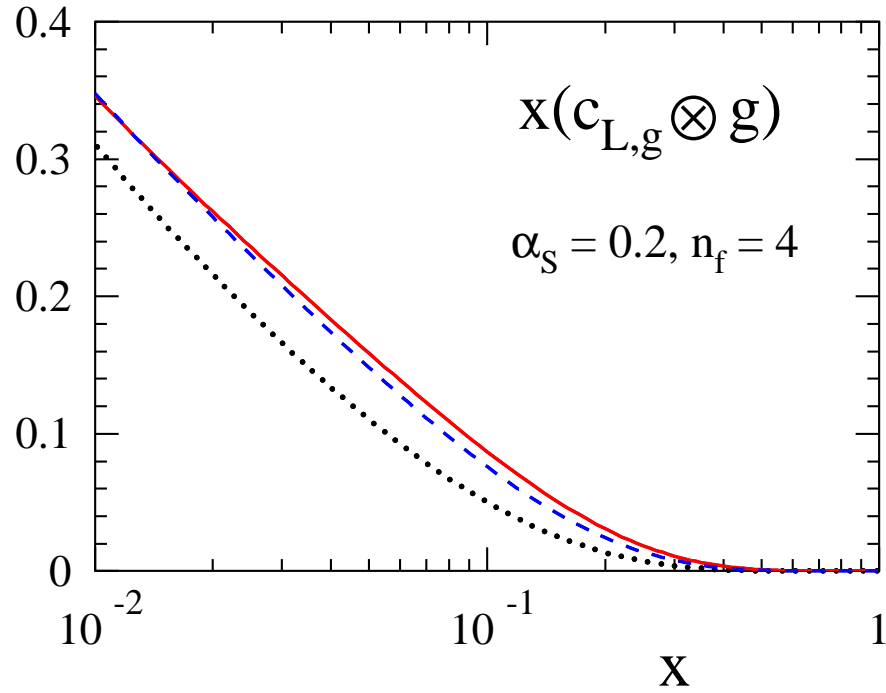
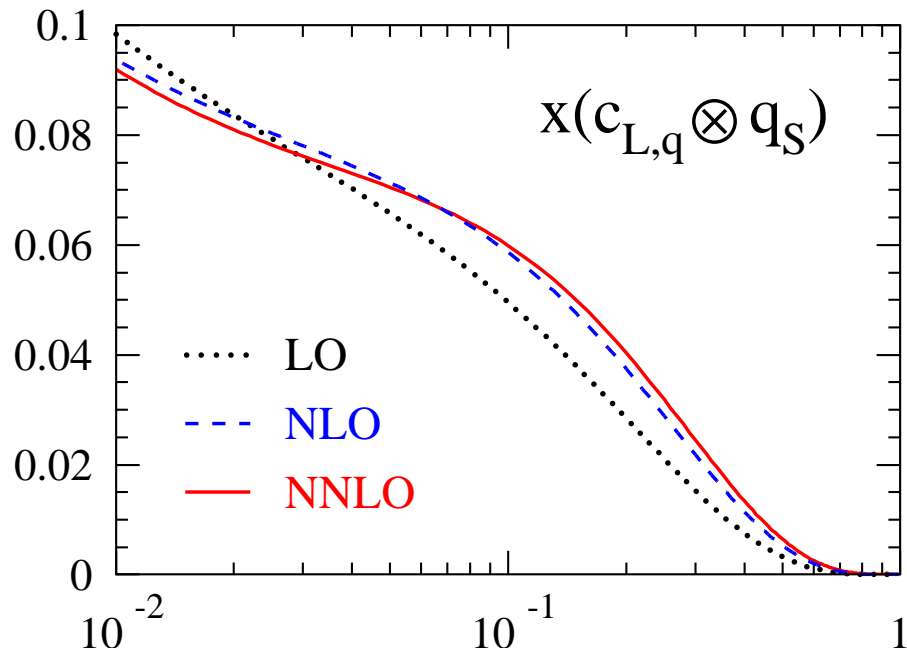
- Perturbative expansion of singlet-quark and gluon contributions to F_L for $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$ (results divided by $\langle e^2 \rangle$)
- Parametrization of singlet distributions (order independent)

$$xq_s(x, \mu_0^2) =$$

$$0.6 x^{-0.3} (1-x)^{3.5} (1 + 5.0 x^{0.8})$$

$$xg(x, \mu_0^2) =$$

$$1.6 x^{-0.3} (1-x)^{4.5} (1 - 0.6 x^{0.3})$$

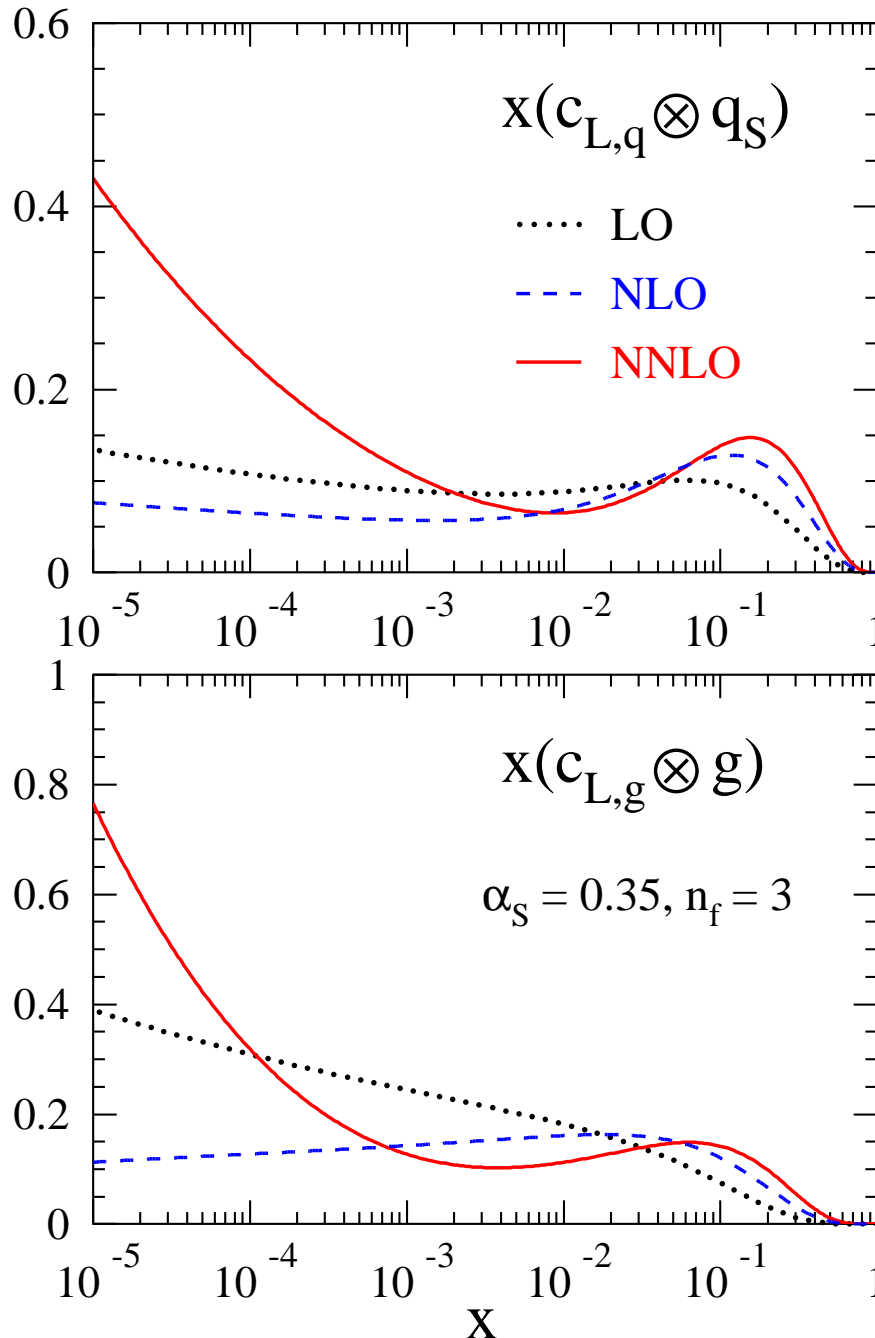


- Perturbative expansion of singlet-quark and gluon contributions to F_L for $n_f = 4$ and $\alpha_s(\mu^2) = 0.2$ (results divided by $\langle e^2 \rangle$)
- Parametrization of singlet distributions (order independent)

$$xq_s(x, \mu_0^2) = 0.6 x^{-0.3} (1-x)^{3.5} (1 + 5.0 x^{0.8})$$

$$xg(x, \mu_0^2) = 1.6 x^{-0.3} (1-x)^{4.5} (1 - 0.6 x^{0.3})$$

Zooming in at large x



- Perturbative expansion of singlet-quark and gluon contributions to F_L for $n_f = 3$ and $\alpha_s(\mu^2) = 0.35$ (results divided by $\langle e^2 \rangle$)
- Parametrization choice (order independent) similar to CTEQ5M and LesHouches benchmark
Vogt, Salam '02

$$xq_s(x, \mu_0^2) = 0.6 x^{-0.1} (1-x)^3 (1 + 10 x^{0.8})$$

$$xg(x, \mu_0^2) = 1.2 x^{-0.1} (1-x)^4 (1 + 1.5 x)$$

Sensitivity at low scales $Q^2 \simeq 2 \text{ GeV}^2$

Summary

- Complete QCD results at NNLO for deep-inelastic scattering
- Coefficient functions for F_L at third order
 - first complete three-loop calculation for single-scale hard scattering process
- Precision predictions ready for measurement of F_L at HERA
- Important constraint on gluon distribution through F_L