Infrared Finite Scattering Amplitudes

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Overview

- A *very* brief overview of calculating physical observables and scattering amplitudes.
- Highlight the source of infrared (IR) singularities.
- Discuss ways of avoiding such IR singularities, **IR finite scattering amplitudes**.
- An example demonstrating **IR finite scattering amplitudes** in the calculation of the total cross section for $e^+e^- \longrightarrow$ 2 Jets at NLO.

Calculation of physical observables

- The aim of theoretical calculations is to produce predictions of the **physical observables** measured in experiments.
- This is achieved by integrating over the *phase space* with the amplitude “squared” times some definition function for the *physical observable* being calculated,

$$\int dLips|A|^2 \times J(k_i, ...)$$

- The important part here is the matrix element A, this is calculated from the S**-Matrix**,

$$A = \langle out|S|in\rangle$$

- The S-Matrix maps the basis of the initial states onto the final states and contains all the details of the **interactions**.

The S-Matrix

- Scattering calculations are performed by calculating the overlap of an *initial* and *final* eigenstate of the **full** Hamiltonian, H,

$$\langle \Psi_{out}(\infty)|\Psi_{in}(-\infty)\rangle = \langle \Psi_{out}|U(\infty,t')U^{\dagger}(-\infty,t')|\Psi_{in}\rangle = \langle \Psi_{out}|S|\Psi_{in}\rangle$$

- To calculate this we usually place it in the **interaction picture**,

$$\langle \Psi_{out}|U_{H_0}(\infty,t')U^{\dagger}_{H_0}(\infty,t')SU_{H_0}(-\infty,t')U^{\dagger}_{H_0}(-\infty,t')|\Psi_{in}\rangle$$
$$= \langle \psi^{I}_{out}|S^{I}|\psi^{I}_{in}\rangle$$

- The *in* and *out* fields are now eigenstates of the **free** Hamiltonian, H_0.
- The S-Matrix now evolves in time with the **interaction** Hamiltonian, $H_I = H - H_0$.
- For this transformation to be valid the **interaction** must *"turn off"* at infinity.

Infrared divergences

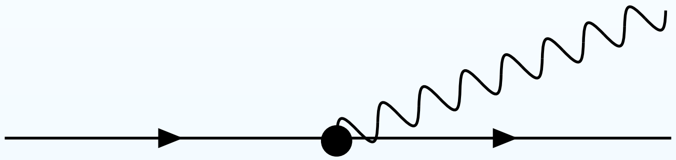
- The S-Matrix is then expanded perturbatively in terms of some parameter α.
- At each order in the **time ordered** perturbative expansion we have terms of the form,

$$\alpha^n \int_{-\infty}^{\infty} dt_1 H_I e^{-it_1(E_f - E_{a_1})} \ldots \int_{-\infty}^{t_{n-1}} dt_n H_I e^{-it_n(E_{a_n} - E_i)}$$

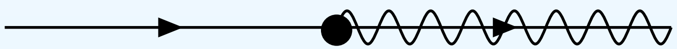
- For the perturbative expansion to make sense these *time* integrals must **converge**. [P.P.Kulish & L.D.Faddeev Theor.Math.Phys. **4**, 745 (1970)]
- For the interaction to *"turn off"* we require $\exp(-it_1(E_f - E_{a_1}))$ to vanish at $t_1 \to \pm\infty$, this is usually achieved by using an *adiabatic* factor.
- We can quickly see though that if $E_f - E_{a_1} = 0$, then $\exp(-it_1(E_f - E_{a_1})) \to 1$ and this integral will in fact **diverge**.

The appearance of infrared divergences

- Situations where $E_f - E_{a_1} = 0$ can occur are,
  - **Soft** emission, $E_{p-k} + E_k - E_p \to 0$ as $E_k \to 0$,

$E_{a_1} = E_p$  $E_f = E_k + E_{p-k}$

  - **Collinear** emission,
    $E_{p-k} + E_k - E_p \to (1-\lambda)E_p + \lambda E_p - E_p = 0$ as $E_k \to \lambda E_p$,

$E_{a_1} = E_p$  $E_f = E_k + E_{p-k}$

- These divergences will **only** appear when the particles involved are *“on-shell”*, so **IR divergences** will only appear in **initial** and **final** states (as internal particles are *“off-shell”*).
- *States* at infinity are **not** free and well separated. The interaction does **not** *“turn off”*.

So whats going on?

- The problem is that the energy of the states $|p,k\rangle_{E_k \to 0, E_k \to \lambda E_p}$ is **degenerate** with those of $|p\rangle$.
- By assuming that the *in* and *out* states are eigenstates of H_0 we mistakingly split up states that *“look”* the same.
- A possible **solution** is to use **asymptotic states** that are eigenstates of the **asymptotic** Hamiltonian, $H_A = H_0 + H_{\text{IR}}$, these *combine* H_0 states with the same energy into *one* state.

$$|\{p\}\rangle \equiv$$

- We can do this by switching to the **asymptotic interaction picture**. The *in* and *out* states then become **asymptotic** states.
- **Energy** is now an eigenvalue of the **asymptotic** Hamiltonian H_A and $E_f - E_{a_1} \neq 0$ as the state $\langle\{f\}| \neq \langle\{a_1\}|$, leading to a finite result as **degenerate** states are **no longer separate**.

Infrared finite scattering amplitudes

- **Problem!** this is *very hard* to do. We need to solve for the eigenstates of H_A, which we do **not** know how to do.
- Instead, perform a different **unitary** transformation on the S-Matrix. Changing into the **asymptotic interaction** picture involves,

$$A = \langle \psi^I_{out} | U_{H_A - H_0} U^\dagger_{H_A - H_0} S^I U_{H_A - H_0} U^\dagger_{H_A - H_0} | \psi^I_{in} \rangle = \langle \psi^A_{out} | S^A_A | \psi^A_{in} \rangle$$

Instead **remain** in the **interaction picture** but alter the S-Matrix,

$$\mathcal{A} = \langle \psi^I_{out} | U^\dagger_{H_A - H_0} S^I U_{H_A - H_0} | \psi^I_{in} \rangle = \langle \psi^I_{out} | S^I_A | \psi^I_{in} \rangle$$

- *Physical observables* remain **unaltered** by this transformation as $\int |A|^2 \equiv \int |\mathcal{A}|^2$, only the amplitudes **change**.
- The amplitudes, $\mathcal{A}$, are called **infrared finite scattering amplitudes** and it is known that these are are *also* **free** of **IR singularities.** [J.Frenkel et.al. Nucl. Phys. B**194**, 172 (1982) and others]

How does this work?

- **IR finite scattering amplitudes** differ from the usual approach by the basis of states used,
  - The S**-Matrix** usually acts on the **complete** basis of H_0 states including the **degenerate** states.
  - The S_A**-Matrix** acts on a **reduced** basis of H_0 states that does **not** include any of the **soft** or **collinear** states. So the amplitude is finite.
- So we can use our knowledge of **energy** eigenstates of H_0 to produce amplitudes that are **free** of IR divergences.
- There is one problem, unlike in the usual case where $[S, H_0] = 0$, **IR finite scattering amplitudes** use S_A and $[S_A, H_0] \neq 0$ and so we do not expect the amplitude to contain a single energy conserving delta function.

An example: $e^+e^- \longrightarrow$ 2 Jets at NLO

- Check that **IR finite scattering amplitudes** give the correct answer for $e^+e^- \to$ 2 Jets at NLO. [Forde & Signer Nucl. Phys. B**684** 125, (2004), arXiv:hep-ph/0311059]
- We choose a definition of S_A that **removes** all the *singularities* in the three point vertex. This once chosen is **general** for **all** processes involving three point vertices.
- There are **two** amplitudes which will contribute to this process.
  - $\mathcal{A}(\{q(p_1),\bar{q}(p_2)\};\gamma)$, which has one incoming photon and two outgoing *"quarks"*.
  - $\mathcal{A}(\{q(p_1),\bar{q}(p_2),g(p_3)\};\gamma)$, has **one** incoming **photon**, two outgoing *"quarks"* and one outgoing *"gluon"*.
- They will both be IR finite unlike the usual technique.

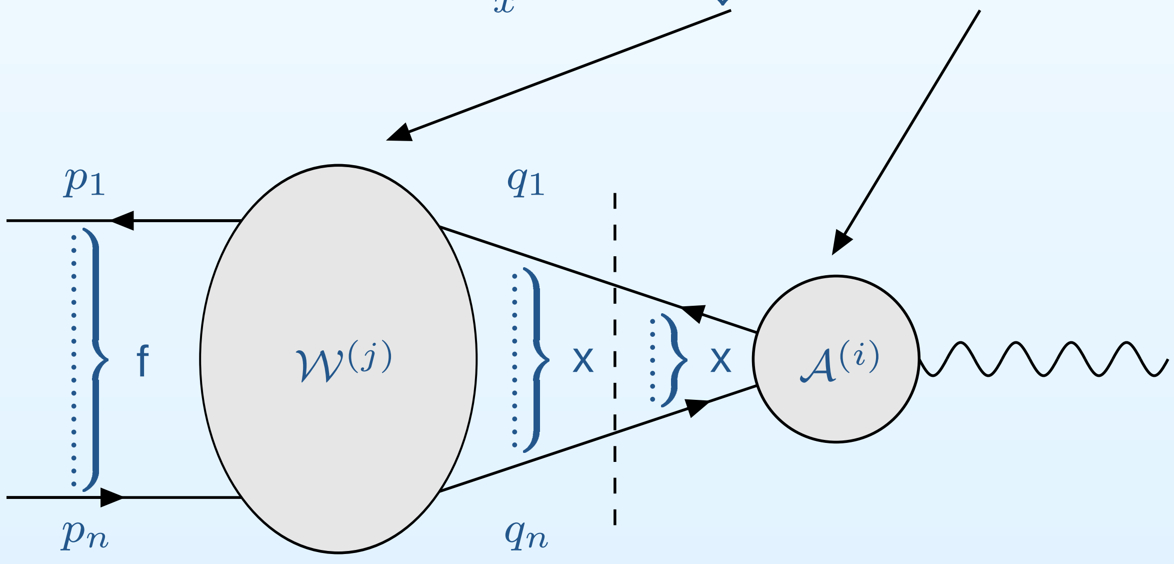
An example: $e^+e^- \longrightarrow$ 2 Jets at NLO

- Unlike the normal situation we do not want to calculate this **directly**, instead we rewrite the amplitude as,

$$\langle f|S_A|i\rangle \equiv \langle f|U^\dagger_{H_A-H_0} S U_{H_A-H_0}|i\rangle = \langle\{f\}|S|\{i\}\rangle$$

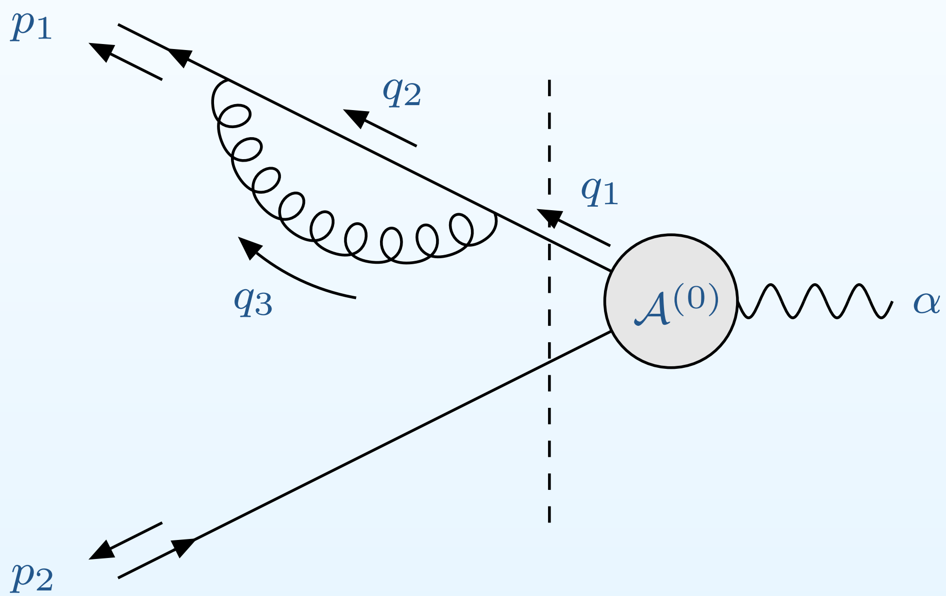
- So the **amplitudes** can be split up into **sub-amplitudes** as,

$$\mathcal{A}(\{f\};\{i\}) = \langle\{f\}|S|\{i\}\rangle = \sum_x \underbrace{\langle f|U^\dagger_{H_A-H_0}|x\rangle}\,\underbrace{\langle x|S|\gamma\rangle}$$



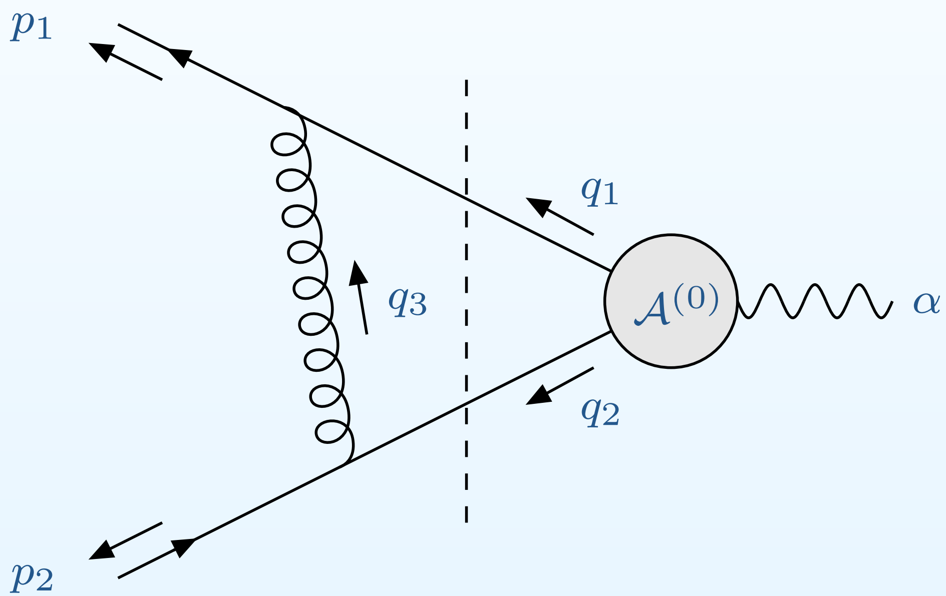
IR free amplitudes at $\mathcal{O}(g^2)$ for $e^+e^- \longrightarrow$ 2 Jets at NLO

- $\mathcal{A}(\{q(p_1), \bar{q}(p_2)\}; \gamma)$ is made up of **seven** *sub-amplitudes*.
- The **first** pair of *sub-amplitudes* is given by the **self-interaction** terms, one for each leg,



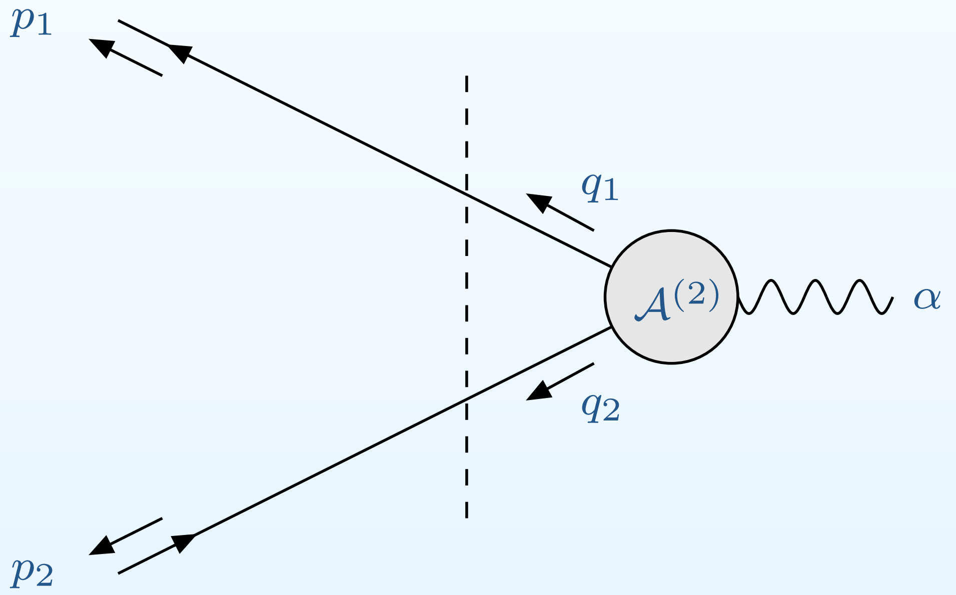
IR free amplitudes at $\mathcal{O}(g^2)$ for $e^+e^- \longrightarrow$ 2 Jets at NLO

- $\mathcal{A}(\{q(p_1), \bar{q}(p_2)\}; \gamma)$ is made up of **seven** *sub-amplitudes*.
- The **second** pair of *sub-amplitudes* is given by the 1-gluon exchange diagrams, one for each time ordering,



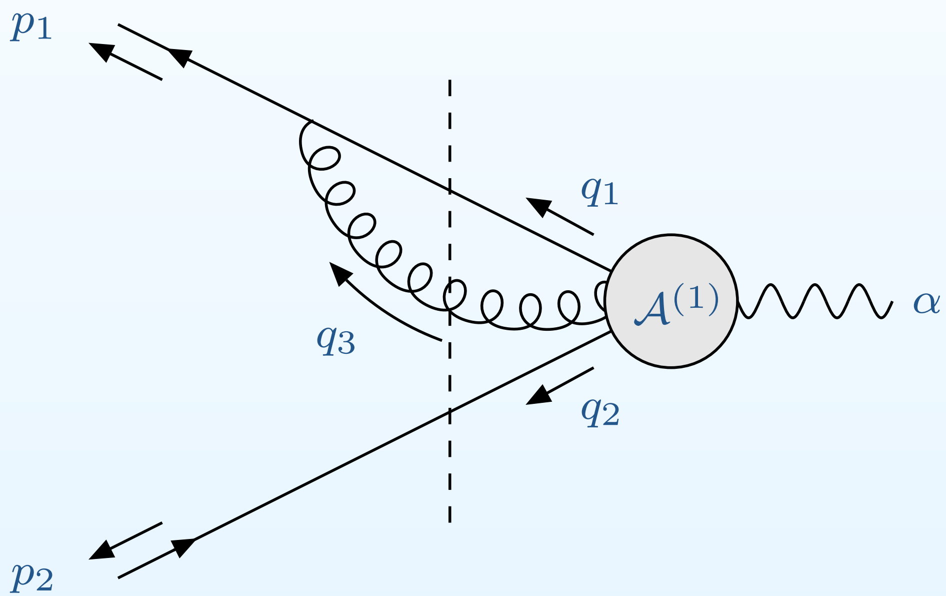
IR free amplitudes at $\mathcal{O}(g^2)$ for $e^+e^- \longrightarrow$ 2 Jets at NLO

- $\mathcal{A}(\{q(p_1), \bar{q}(p_2)\}; \gamma)$ is made up of **seven** *sub-amplitudes*.
- The **fifth** *sub-amplitude* is given by the usual virtual correction term,



IR free amplitudes at $\mathcal{O}(g^2)$ for $e^+e^- \longrightarrow$ 2 Jets at NLO

- $\mathcal{A}(\{q(p_1), \bar{q}(p_2)\}; \gamma)$ is made up of **seven** *sub-amplitudes*.
- The **final** pair of *sub-amplitudes* is given by the two 3-particle cut diagrams,



An IR Finite Amplitude

- We can now **combine** these *sub-amplitudes* together.
- All the infrared singular pieces **cancel** between the *sub-amplitudes*, leaving an **infrared finite amplitude**,

$$
\begin{aligned}
&\mathcal{A}(\{q(p_1),\bar{q}(p_2)\};\gamma) = \\
&\quad \left(1 + C_F\,\left(\frac{\alpha_s}{2\pi}\right)\left(g_1(\Delta) + g_2(\Delta) + g_3(\Delta) - 4 + \frac{\pi^2}{12}\right)\right) \\
&\qquad\qquad\qquad\qquad \mathcal{A}^{(0)}(q(p_1),\bar{q}(p_2);\gamma(P)) \\
&\quad +(-ie)\,\delta_{ij}\,\langle p_1|\gamma^\alpha|p_2\rangle(2\pi)^{(D-1)}\delta^{(D-1)}(\vec{P}-\vec{p}_1-\vec{p}_2) \\
&\quad \int \mathrm{d}\widetilde{q}_3\,\left(f_1(p_1,p_2,q_3)\delta(\sqrt{S}-\omega(\vec{p}_1)-\omega(\vec{p}_2)) + f_2(p_1,p_2,q_3)\right. \\
&\quad \left. +f_3(p_1,p_2,q_3)\right)
\end{aligned}
$$

A Cross Section Calculation

- Using the two **IR finite amplitudes**, $\mathcal{A}(\{q(p_1),\bar{q}(p_2)\};\gamma)$ and $\mathcal{A}(\{q(p_1),\bar{q}(p_2),g(p_3)\};\gamma)$, we can calculate the **total cross section** for $e^+e^- \to$ 2 Jets at NLO.
- The cross sections are,

$$\begin{aligned}
\sigma_{\{q\bar{q}\}} &= \int |\mathcal{A}(\{q(p_1),\bar{q}(p_2)\};\gamma)|^2 \\
&= \left(1 + C_F\frac{\alpha_s}{\pi}\left(-\frac{1}{2} + \log 4 - \frac{3}{2}\log\left(\frac{\Delta}{2}\right) - \log^2\left(\frac{\Delta}{2}\right)\right)\right), \\
\sigma_{\{q\bar{q}g\}} &= \int |\mathcal{A}(\{q(p_1),\bar{q}(p_2),g(p_3)\};\gamma)|^2 \\
&= C_F\left(\frac{\alpha_s}{\pi}\right)\left(\frac{5}{4} - \log 4 + \frac{3}{2}\log\left(\frac{\Delta}{2}\right) + \log^2\left(\frac{\Delta}{2}\right)\right).
\end{aligned}$$

- Adding these two pieces together gives,

$$\frac{1}{\sigma_0}\sigma = \frac{1}{\sigma_0}\left(\sigma_{\{q\bar{q}\}} + \sigma_{\{q\bar{q}g\}}\right) = \left(1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right)$$

the same as the usual result given by $\sigma = \sigma_{q\bar{q}} + \sigma_{q\bar{q}g}$.

Conclusion

- Given a *very brief* overview of infrared singularities
- Shown how **IR finite amplitudes** can be produced.
- Shown how this technique can be **applied** to an example process, $e^+e^- \rightarrow$ 2 Jets at NLO, producing the **known** result but using amplitudes that are **free** of IR singularities.
- Future work,
  - Develop the technique for **direct** calculations of **IR finite amplitudes** without the intermediate sub-amplitudes.
  - Develop techniques to calculate *physical observables* from these amplitudes.
  - ...