Basis-independent description of CP-violation in the Two-Higgs-Doublet Model

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This talk is based on work that appears in:

1. G. Branco, L. Lavoura and J.P. Silva, *CP Violation* (Oxford University Press, Oxford, England, 1999), chapters 22 and 23.

2. S. Davidson and H.E. Haber, "Basis-independent methods for the two-Higgs-doublet model," SCIPP-04/15 (hep-ph/0504nnn).

3. J.F. Gunion and H.E. Haber, "Conditions for explicit CP-Violation in the general two-Higgs-doublet model," SCIPP-04/16 (hep-ph/0504nnn).

Outline

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Motivation

The Higgs sector of the minimal supersymmetric extension of the Standard Model (MSSM) is a constrained two-Higgs-doublet model (2HDM). However, at one-loop all possible 2HDM interactions allowed by gauge invariance are generated (due to SUSY-breaking interactions).

Thus, the Higgs sector of the MSSM is in reality the most general (CP-violating) 2HDM model—albeit with certain relations among the Higgs sector parameters determined by the fundamental parameters of the broken supersymmetric model.

The general 2HDM consists of two identical (hypercharge-one) scalar doublets Φ_1 and Φ_2. To determine the physical quantities of the theory, one must develop basis-independent techniques.

Questions:

- Is the Higgs sector CP-violating?
- If yes, is the CP-violation explicit or spontaneous?

One can always arrange the vacuum expectation values (vevs) of the Higgs field to take the form:

$$\langle\Phi_1\rangle = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 0 \\ v_1 \end{array}\right), \qquad \langle\Phi_2\rangle = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 0 \\ v_2\, e^{i\xi} \end{array}\right),$$

where v_1 and v_2 are real and non-negative, $0 \leq \xi < 2\pi$ and $v^2 \equiv v_1^2 + v_2^2 = 4m_W^2/g^2 = (246\ \mathrm{GeV})^2$.

But a further phase redefinition $\Phi_2 \to e^{i\xi}\Phi_2$ removes the phase from the vevs. So, how can one really be sure about the nature of Higgs-mediated CP-violation?

Compare this situation with broken global symmetries. The existence or non-existence of mass for a (would-be) Goldstone boson provides the evidence for or against a spontaneously broken global symmetry.

The General Two-Higgs-Doublet Model

Consider the 2HDM potential in a *generic* basis:

$$\mathcal{V} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}] + \tfrac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2$$

$$+ \tfrac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)$$

$$+ \left\{ \tfrac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \left[\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)\right] \Phi_1^\dagger \Phi_2 + \text{h.c.} \right\}$$

A basis change consists of a U(2) transformation $\Phi_a \longrightarrow U_{a\bar{b}} \Phi_b$ (and $\Phi_{\bar{a}}^\dagger \longrightarrow \Phi_{\bar{b}}^\dagger U_{b\bar{a}}^\dagger$). Here, U(2) $\cong$ SU(2)$\times$U(1)$_{\rm Y}$. The parameters m_{11}^2, m_{22}^2, m_{12}^2, and $\lambda_1, \ldots, \lambda_7$ are transformed under the "flavor"-SU(2) transformation. To identify invariants, write :

$$\mathcal{V} = Y_{a\bar{b}} \Phi_{\bar{a}}^\dagger \Phi_b + \tfrac{1}{2} Z_{a\bar{b}c\bar{d}} (\Phi_{\bar{a}}^\dagger \Phi_b)(\Phi_{\bar{c}}^\dagger \Phi_d)\,,$$

where $Z_{a\bar{b}c\bar{d}} = Z_{c\bar{d}a\bar{b}}$ and hermiticity implies

$$Y_{a\bar{b}} = (Y_{b\bar{a}})^*\,, \qquad Z_{a\bar{b}c\bar{d}} = (Z_{b\bar{a}d\bar{c}})^*\,.$$

The barred indicies help keep track of which indices transform with U and which transform with $U^\dagger$. For example, $Y_{a\bar{b}} \longrightarrow U_{a\bar{c}} Y_{c\bar{d}} U_{d\bar{b}}^\dagger$ and $Z_{a\bar{b}c\bar{d}} \rightarrow U_{a\bar{e}} U_{f\bar{b}}^\dagger U_{c\bar{g}} U_{h\bar{d}}^\dagger Z_{e\bar{f}g\bar{h}}$.

Conditions for explicit CP-violation

Here, we consider the conditions for Higgs-mediated CP-violation due to an explicitly CP-violating Higgs potential.*

Theorem 1: The Higgs potential is CP-conserving if and only if there exists a basis in which all Higgs potential parameters are real.

Potentially complex Higgs potential parameters are: m_{12}^2, λ_5, λ_6 and λ_7. Of course, these are basis-dependent quantities. Nevertheless, the following result should be noted:

Theorem 2: In a generic basis, the following is a sufficient (but not a necessary) condition for an explicitly CP-conserving 2HDM scalar potential:

$$\begin{aligned}\mathrm{Im}\,([m_{12}^2]^2\lambda_5^*) &= \mathrm{Im}\,(m_{12}^2\lambda_6^*) = \mathrm{Im}\,(m_{12}^2\lambda_7^*) \\ &= \mathrm{Im}\,(\lambda_5^*\lambda_6^2) = \mathrm{Im}\,(\lambda_5^*\lambda_7^2) = \mathrm{Im}\,(\lambda_6^*\lambda_7) = 0\,.\end{aligned}$$

*We defer the question of whether CP is spontaneously broken if the Higgs potential is manifestly CP-conserving.

Clearly, the latter is not good enough. We shall instead provide a set of basis-independent conditions. The complete set of conditions is summarized by the following result:

Theorem 3: The following are the necessary and sufficient conditions for an explicitly CP-conserving 2HDM scalar potential:

$$I_{3Y3Z} = 0\,, \qquad \text{if } \lambda_1 = \lambda_2 \textit{ and } \lambda_6 = -\lambda_7$$
$$I_{Y3Z} = I_{2Y2Z} = I_{6Z} = 0\,, \qquad \text{otherwise}$$

where

$$\begin{aligned}
I_{Y3Z} &\equiv \mathrm{Im}(Y_{d\bar{a}} Z^{(1)}_{a\bar{c}} Z^{(1)}_{e\bar{b}} Z_{b\bar{e}c\bar{d}})\,,\\
I_{2Y2Z} &\equiv \mathrm{Im}(Y_{a\bar{b}} Y_{c\bar{d}} Z_{b\bar{a}d\bar{f}} Z^{(1)}_{f\bar{c}})\,,\\
I_{6Z} &\equiv \mathrm{Im}(Z_{a\bar{b}c\bar{d}} Z^{(1)}_{b\bar{f}} Z^{(1)}_{d\bar{h}} Z_{f\bar{a}j\bar{k}} Z_{k\bar{j}m\bar{n}} Z_{n\bar{m}h\bar{c}})\,,\\
I_{3Y3Z} &\equiv \mathrm{Im}(Y_{q\bar{f}} Y_{h\bar{b}} Y_{g\bar{a}} Z_{e\bar{h}f\bar{q}} Z_{c\bar{e}d\bar{g}} Z_{a\bar{c}b\bar{d}})\,.
\end{aligned}$$

Above, we have introduced:

$$Z^{(1)}_{a\bar{d}} \equiv \delta_{b\bar{c}} Z_{a\bar{b}c\bar{d}} = Z_{a\bar{b}b\bar{d}}\,.$$

Explicit results

$$
\begin{aligned}
I_{6Z} &= 2|\lambda_5|^2 \mathrm{Im}[(\lambda_7^*\lambda_6)^2] - \mathrm{Im}[\lambda_5^{*\,2}(\lambda_6-\lambda_7)(\lambda_6+\lambda_7)^3] \\
&+2\mathrm{Im}(\lambda_7^*\lambda_6)\left[|\lambda_5|^2[|\lambda_6|^2+|\lambda_7|^2-(\lambda_1-\lambda_2)^2]-2(|\lambda_6|^2-|\lambda_7|^2)^2\right] \\
&+(\lambda_1-\lambda_2)\mathrm{Im}\Big[[\Lambda^*-2\lambda_5^*(\lambda_6+\lambda_7)](\lambda_7-\lambda_6)(\lambda_7^*\lambda_6-\lambda_6^*\lambda_7)\Big] \\
&-(\lambda_1-\lambda_2)\mathrm{Im}(\lambda_5^*\Lambda^2)-2(|\lambda_6|^2-|\lambda_7|^2)\mathrm{Im}[\lambda_5^*\Lambda(\lambda_6+\lambda_7)] \\
&+(\lambda_1-\lambda_2)|\lambda_5|^2\mathrm{Im}[\lambda_5^*(\lambda_6+\lambda_7)^2]\,,
\end{aligned}
$$

$$
\begin{aligned}
I_{Y3Z} &= 2(|\lambda_6|^2-|\lambda_7|^2)\mathrm{Im}[Y_{12}(\lambda_6^*+\lambda_7^*)] \\
&+(Y_{11}-Y_{22})\Big[\mathrm{Im}[\lambda_5^*(\lambda_6+\lambda_7)^2]-(\lambda_1-\lambda_2)\mathrm{Im}(\lambda_7^*\lambda_6)\Big] \\
&+(\lambda_1-\lambda_2)\Big[\mathrm{Im}(Y_{12}\Lambda^*)-\mathrm{Im}[Y_{12}\lambda_5^*(\lambda_6+\lambda_7)]\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
I_{2Y2Z} &= (\lambda_1-\lambda_2)\mathrm{Im}(Y_{12}^2\lambda_5^*)-\mathrm{Im}[(Y_{12}\lambda_6^*)^2] \\
&+\mathrm{Im}[(Y_{12}\lambda_7^*)^2]+[(Y_{11}-Y_{22})^2-2|Y_{12}|^2]\mathrm{Im}(\lambda_7^*\lambda_6) \\
&-(Y_{11}-Y_{22})\left[\mathrm{Im}(Y_{12}\Lambda^*)+\mathrm{Im}(Y_{12}\lambda_5^*(\lambda_6+\lambda_7))\right]\,,
\end{aligned}
$$

where

$$\Lambda \equiv (\lambda_2-\lambda_3-\lambda_4)\lambda_6+(\lambda_1-\lambda_3-\lambda_4)\lambda_7\,.$$

The expression for I_{3Y3Z} is very long and will not be given here.

Enumerating all possible invariants

An arbitrary invariant is a product of Y's and Z's, where all possible indices are tied together (*i.e.*, summing unbarred/barred indices in all possible ways). That is,

$$J \equiv Z_{a\bar{a}'b\bar{b}'} Z_{c\bar{c}'d\bar{d}'} \cdots Y_{g\bar{g}'} Y_{h\bar{h}'} \cdots ,$$

where $\{a', b', c', d', \ldots, g', h', \ldots\}$ is a permutation of $\{a, b, c, d, \ldots, g, h, \ldots\}$. If the invariant J contains n_Z factors of Z and n_Y factors of Y, then there are $(2n_Z + n_Y)!$ possible invariants of order $(n_Z + n_Y)$. We wish to determine whether $I \equiv \text{Im}\, J \neq 0$, and how many of these are independent.†

Invariant type	number
5Z or 2Y4Z	3,628,800
Y4Z or 3Y3Z	362,880
2Y3Z	40,320
6Z	479,001,600
Y5Z	39,916,800

†The imaginary parts of many of these invariants trivially reduce to lower order ones, if one sums over the indices of a given Z (which can produce, *e.g.*, $\text{Tr}\, Z^{(1)} = \lambda_1 + \lambda_2 + 2\lambda_4$) or a given Y (which can produce $\text{Tr}\, Y = Y_{11} + Y_{22}$).

Based on analytic work and exploration via Mathematica:

- All invariants of cubic order or less are manifestly real.

- The imaginary part of any potentially complex quartic invariant is a real linear combination of I_{Y3Z} and I_{2Y2Z}.

- The imaginary part of any potentially complex fifth-order invariant vanishes if $I_{Y3Z} = I_{2Y2Z} = 0$.

- The imaginary part of any potentially complex sixth-order invariant that is independent of Y is proportional to I_{6Z}. Moreover, if $Y_{a\bar{b}} = 0$ then the imaginary part of any invariant of arbitrary order vanishes if $I_{6Z} = 0$.

- The imaginary part of any potentially complex sixth order invariant that is both cubic in Y and Z respectively is a real linear combination of the invariant I_{3Y3Z} and lower-order invariants that vanish if $I_{Y3Z} = I_{2Y2Z} = 0$.

- The imaginary part of any invariant of arbitrary order vanishes if $I_{Y3Z} = I_{2Y2Z} = I_{6Z} = I_{3Y3Z} = 0$.

To see that all four invariants introduced above are required, we first note that there always exists a basis in which $\lambda_7 = -\lambda_6$. [Proof: noting that

$$Z^{(1)} = \begin{pmatrix} \lambda_1 + \lambda_4 & \lambda_6 + \lambda_7 \\ \lambda_6^* + \lambda_7^* & \lambda_2 + \lambda_4 \end{pmatrix}$$

is an hermitian matrix, we can always diagonalize it.] In the $\lambda_7 = -\lambda_6$ basis (this basis is not unique),

$$I_{6Z} = -(\lambda_1 - \lambda_2)^3 \,\mathrm{Im}(\lambda_5^* \lambda_6^2)\,,$$

$$I_{Y3Z} = -(\lambda_1 - \lambda_2)^2 \,\mathrm{Im}(Y_{12} \lambda_6^*)\,,$$

$$I_{2Y2Z} = (\lambda_1 - \lambda_2)\left[\mathrm{Im}(Y_{12}^2 \lambda_5^*) + (Y_{11} - Y_{22})\mathrm{Im}(Y_{12}\lambda_6^*)\right]\,.$$

First, suppose that $\lambda_1 \neq \lambda_2$. Then consider three cases:

1. $Y_{a\bar{b}} = 0$ $\quad [\Longrightarrow I_{Y3Z} = I_{2Y2Z} = I_{3Y3Z} = 0]$
2. $\lambda_6 = 0$ and $Y_{11} = Y_{22}$ $\quad [\Longrightarrow I_{6Z} = I_{Y3Z} = I_{3Y3Z} = 0]$
3. $\lambda_5 = Y_{11} = Y_{22} = 0$ and $\mathrm{Re}(Y_{12}\lambda_6^*) = 0$
   $[\Longrightarrow I_{6Z} = I_{2Y2Z} = I_{3Y3Z} = 0]$.

In each case there is only one potentially complex invariant.

In a basis where $\lambda_6 = -\lambda_7$,

$$
\begin{aligned}
I_{3Y3Z} = {} & 2\,\mathrm{Im}(Y_{12}^3 \lambda_6 (\lambda_5^*)^2) - 4\,\mathrm{Im}(Y_{12}^3 (\lambda_6^*)^3) \\
& + [(Y_{11} - Y_{22})^2 - 6|Y_{12}|^2](Y_{11} - Y_{22})\,\mathrm{Im}(\lambda_6^2 \lambda_5^*) \\
& - (\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4)\Big\{(Y_{11} - Y_{22})\,\mathrm{Im}(Y_{12}^2 (\lambda_6^*)^2) \\
& \qquad - \mathrm{Im}(Y_{12}^3 \lambda_5^* \lambda_6^*) + \left[(Y_{11} - Y_{22})^2 - |Y_{12}|^2\right] \mathrm{Im}(Y_{12} \lambda_6 \lambda_5^*)\Big\} \\
& + \Big\{(4|\lambda_6|^2 - 2|\lambda_5|^2)\left[(Y_{11} - Y_{22})^2 - |Y_{12}|^2\right] \\
& \qquad + (\lambda_1 - \lambda_2)^2 Y_{11} Y_{22}\Big\}\mathrm{Im}(Y_{12} \lambda_6^*) \\
& + \left[(\lambda_1 - \lambda_3 - \lambda_4)(\lambda_2 - \lambda_3 - \lambda_4) + 2|\lambda_6|^2 - |\lambda_5|^2\right] \\
& \qquad \times (Y_{11} - Y_{22})\,\mathrm{Im}(Y_{12}^2 \lambda_5^*)\,.
\end{aligned}
$$

If $\lambda_6 = 0$ and $Y_{11} = Y_{22}$, then $I_{3Y3Z} = 0$. In this case, only I_{YYZZ} is potentially complex.

If $\lambda_5 = Y_{11} = Y_{22} = 0$ and $\mathrm{Re}(Y_{12}\lambda_6^*) = 0$, then $I_{3Y3Z} = 0$ and I_{Y3Z} is potentially complex.

If $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$, then $I_{6Z} = I_{Y3Z} = I_{2Y2Z} = 0$. Nevertheless, CP can still be violated if $I_{3Y3Z} \neq 0$.

Dependant invariants

Here are some examples of "new invariants" that are not independent of the four invariants previously identified. Consider:

$$I_{2Y3Z} \equiv \text{Im}(Z_{a\bar{c}b\bar{e}} Z_{c\bar{f}d\bar{b}} Z_{e\bar{g}f\bar{h}} Y_{g\bar{a}} Y_{h\bar{d}}) \,,$$

$$I_{Y4Z} \equiv \text{Im}\,(Z^{(2)}_{a\bar{b}} Z_{b\bar{a}c\bar{d}} Z^{(2)}_{d\bar{e}} Z_{e\bar{c}f\bar{g}} Y_{g\bar{f}}) \,,$$

where $Z^{(2)}_{c\bar{d}} \equiv \delta_{b\bar{a}} Z_{a\bar{b}c\bar{d}} = Z_{a\bar{a}c\bar{d}}$. Then, in the $\lambda_7 = -\lambda_6$ basis,

$$\begin{aligned} I_{2Y3Z} = -2\lambda_4\; I_{2Y2Z} + (\lambda_1 - \lambda_2) \Big[& 4\;\text{Im}\;(Y_{12}^2 \lambda_6^{*2}) \\ & +2\;(Y_{11} - Y_{22})\;\text{Im}\;(Y_{12}\lambda_5^*\lambda_6) \\ & +(\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4)\;\text{Im}\;(Y_{12}^2\lambda_5^*) \Big] \,, \end{aligned}$$

$$I_{Y4Z} = -\lambda_4\; I_{Y3Z} + (\lambda_1 - \lambda_2)^2\;\text{Im}\;(Y_{12}\lambda_5^*\lambda_6) \,.$$

Noting that $\text{Im}\;(Y_{12}^2\lambda_6^{*2}) = 2\;\text{Im}\;(Y_{12}\lambda_6^*)\;\text{Re}\;(Y_{12}\lambda_6^*)$, *etc.*, it is easy to show that if $I_{2Y2Z} = I_{Y3Z} = 0$ (in the $\lambda_7 = -\lambda_6$ basis), then $I_{2Y3Z} = I_{Y4Z} = 0$. But these are invariant (basis-independent) quantities, so this result must be true in any basis.

Thus, $I_{2Y2Z} = I_{Y3Z} = 0$ is sufficient to guarantee that all potentially complex invariants of order five or less are all real.

Special model cases:

1. $\lambda_1 = \lambda_2, \quad \lambda_6 = \lambda_7, \quad Y_{11} = Y_{22}$, where Y_{12}, λ_5 and λ_6 have arbitrary phases.

2. $\lambda_1 + \lambda_2 = 2(\lambda_3 + \lambda_4), \quad \lambda_5 = 0, \quad \lambda_6 = \lambda_7$, where Y_{12} and λ_6 have arbitrary phases.

3. $\lambda_1 = \lambda_2, \quad \lambda_6 = \lambda_7^*, \quad Y_{11} = Y_{22}$, and Y_{12} and λ_5 are real.

All four CP-odd invariants vanish for these three models. Thus, these models explicitly conserve CP (despite the fact that the conditions of Theroem 2 are not necessarily satisfied).

Model 3 arises by imposing a discrete permutation symmetry, $\Phi_1 \leftrightarrow \Phi_2$. If in addition λ_6 is real, then there exists a minimum of the scalar potential with $v_1 = v_2$ and $\xi \neq 0$. Nevertheless, CP is not spontaneously broken, since one can find a U(2) transformation to a new basis in which all scalar potential parameters are real and $\xi = 0 \pmod{\pi}$.

Proof of Theorem 3

Suppose there exist a basis in which Y_{12} and $\lambda_{5,6,7}$ are real. Then, in this basis, the imaginary part of any invariant vanishes. But, invariants are basis-independent. Thus, all invariant quantities made up of the $Y_{a\bar{b}}$ and the $Z_{a\bar{b}c\bar{d}}$ are real.

The proof of the converse is more involved. We proceed in four steps. Go to the $\lambda_7 = -\lambda_6$ basis.

1. Suppose that $\lambda_1 \neq \lambda_2$. If $I_{2Y2Z} = I_{Y3Z} = I_{6Z} = 0$, then

$$\mathrm{Im}\,(\lambda_5^* \lambda_6^2) = \mathrm{Im}\,(Y_{12}\lambda_6^*) = \mathrm{Im}\,(Y_{12}^2 \lambda_5^*) = 0\,.$$

One can always make a phase rotation $\Phi_1^\dagger \Phi_2 \to e^{i\gamma}\Phi_1^\dagger \Phi_2$ so that λ_6 is real. If $\lambda_6 = 0$, then the phase γ will be chosen so that λ_5 is real. Thus, after the phase rotation, we are now in a basis where all Higgs potential parameters are real. (In particular, it follows that $I_{3Y3Z} = 0$ as well.)

2. Suppose that $\lambda_1 \neq \lambda_2$ and $Y_{a\bar{b}} = 0$. Then, if $I_{6Z} = 0$, it follows that $\mathrm{Im}\,(\lambda_5^* \lambda_6^2) = 0$. The same phase rotation as above implies that a basis exists where all the λ_i are real.

3. Suppose that $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$. If true in one basis, it is true in all bases. Since $I_{6Z} = 0$, then there exists a basis in which all the λ_i are real.

<u>Proof:</u> If $\lambda_7 = -\lambda_6 = 0$ then one can trivially phase rotate one of the scalar fields such that λ_5 is real. So assume that $\lambda_6 \neq 0$ and phase rotate one of the scalar fields so that λ_6 is real. From this basis, rotate to a new basis with the unitary matrix $U(\theta, \xi, \chi)$:

$$U(\theta, \xi, \chi) = \begin{pmatrix} \cos\theta & e^{-i\xi}\sin\theta \\ -e^{i\chi}\sin\theta & e^{i(\chi-\xi)}\cos\theta \end{pmatrix} .$$

Then,

$$\begin{aligned} \lambda_5' e^{2i\chi} &= \tfrac{1}{4}s_{2\theta}^2\left[\lambda_1 + \lambda_2 - 2\lambda_{345}\right] + \mathrm{Re}(\lambda_5 e^{2i\xi}) + ic_{2\theta}\mathrm{Im}(\lambda_5 e^{2i\xi}) \\ &\quad - s_{2\theta}c_{2\theta}\mathrm{Re}[(\lambda_6 - \lambda_7)e^{i\xi}] - is_{2\theta}\mathrm{Im}[(\lambda_6 - \lambda_7)e^{i\xi})] , \\ \lambda_6' e^{i\chi} &= -\tfrac{1}{2}s_{2\theta}\left[\lambda_1 c_\theta^2 - \lambda_2 s_\theta^2 - \lambda_{345}c_{2\theta} - i\mathrm{Im}(\lambda_5 e^{2i\xi})\right] \\ &\quad + c_\theta c_{3\theta}\mathrm{Re}(\lambda_6 e^{i\xi}) + s_\theta s_{3\theta}\mathrm{Re}(\lambda_7 e^{i\xi}) \\ &\quad + ic_\theta^2\mathrm{Im}(\lambda_6 e^{i\xi}) + is_\theta^2\mathrm{Im}(\lambda_7 e^{i\xi}) . \end{aligned}$$

Put $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$ real in the above equations. Then, look for solutions to $\mathrm{Im}\ \lambda_5' = 0$ and $\mathrm{Im}\ \lambda_6' = 0$. These two equations are equivalent to $\tan 2\chi = 4f_a/f_c$ and $\tan\chi = 2f_d/f_b$, where the f's are functions of θ, ξ and the argument of λ_5. Since $\tan 2\chi = 2\tan\chi/(1-\tan^2\chi)$, it follows that we must find a θ and ξ such that:

$$G(\theta,\xi) \equiv f_a(f_b^2 - 4f_d^2) - f_b f_c f_d = 0\,.$$

Inserting the explicit forms for the f's, one easily shows that $G(\pi/2,\xi) = -G(0,\xi)$, which implies that for any ξ, there exists a θ between 0 and $\pi/2$ that solves $G(\theta,\xi)=0$. That is, we have found the new basis where all the λ_i are real.

4. Suppose that $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$. If $I_{3Y3Z} = 0$, then there exists a basis in which all the Higgs potential parameters are real.

Proof: Using the result of step 3, it follows that one can find a basis where all the λ_i are real. In this basis, $I_{3Y3Z} = 2AB\ \mathrm{Im}\ Y_{12}$, where

$$A \equiv \lambda_5^2 + \lambda_5(\lambda_1 - \lambda_3 - \lambda_4) - 2\lambda_6^2\,,$$

$$B \equiv 4\lambda_6\,(\mathrm{Re}\ Y_{12})^2 - (Y_{11}-Y_{22})(\lambda_3+\lambda_4+\lambda_5-\lambda_1)\,\mathrm{Re}\ Y_{12} - (Y_{11}-Y_{22})^2\lambda_6\,.$$

If $\mathrm{Im}\ Y_{12} = 0$, then our proof is complete. If $\mathrm{Im}\ Y_{12} \neq 0$, one can make a further change of basis to a new basis where $\mathrm{Im}\ Y'_{12} = \mathrm{Im}\ \lambda'_5 = \mathrm{Im}\ \lambda'_6 = 0$ if either $A=0$ or $B=0$.

To prove this assertion requires one to construct the unitary matrix $U(\theta,\xi,\chi)$ such that $\mathrm{Im}\ Y'_{12}=\mathrm{Im}\ \lambda'_5=\mathrm{Im}\ \lambda'_6=0$, where λ'_5 and λ'_6 were given previously and

$$Y'_{12}e^{i\chi}=\tfrac{1}{2}(Y_{22}-Y_{11})s_{2\theta}+\mathrm{Re}(Y_{12}e^{i\xi})c_{2\theta}+i\,\mathrm{Im}(Y_{12}e^{i\xi})\,.$$

We have verified that such a U always exists.[‡] This completes the proof.

[‡]For example, choose $\chi=\pi/2$ and $\xi=-\pi$. Then, $\mathrm{Im}\ \lambda'_5=0$ for arbitrary θ, while $\mathrm{Im}\ Y'_{12}=\mathrm{Im}\ \lambda'_6=0$ requires that θ satisfy:

$$\tan 2\theta=\frac{2|Y_{12}|\cos\theta_{12}}{Y_{22}-Y_{11}}\,,\qquad\qquad \cot 4\theta=\tfrac{1}{4}(\lambda_{345}-\lambda_1)\,,$$

where $\theta_{12}\equiv\ \arg\,Y_{12}$ and $\lambda_{345}\equiv\lambda_3+\lambda_4+\lambda_5$. These two equations for θ are generally inconsistent. But, they are compatible if $B=0$. (Note that $B=0$ is a quadratic equation for $\cos\theta_{12}$.) A different analysis is required if $A=0$.

Conditions for Spontaneous CP-violation

If $I_{6Z} = I_{YZZZ} = I_{YYZZ} = I_{3Y3Z} = 0$, then the Higgs potential is CP-conserving. This means that there exists a family of "real" bases in which all Higgs potential parameters are real. To determine whether CP is spontaneously broken, one must check whether or not the vacuum respects CP.

Theorem 4: Given an explicitly CP-conserving 2HDM potential, CP is spontaneously broken if and only if no real basis can be found in which the Higgs vacuum expectation values are real.

A basis independent formulation of this condition was obtained by Lavoura and Silva and by Botella and Silva. The scalar potential minimum condition is given by:

$$\widehat{v}_{\bar{a}}^{\,*}\left[Y_{a\bar{b}} + \tfrac{1}{2}v^2 Z_{a\bar{b}c\bar{d}}\widehat{v}_{\bar{c}}^{\,*}\widehat{v}_d\right] = 0\,.$$

The most general U(1)$_{\rm EM}$-conserving vev is:

$$\langle\Phi_a\rangle = \frac{v}{\sqrt{2}}\left(\begin{array}{c} 0 \\ \widehat{v}_a\end{array}\right)\,, \qquad \text{with} \qquad \widehat{v}_a \equiv \left(\begin{array}{c} c_\beta \\ s_\beta\, e^{i\xi}\end{array}\right)\,.$$

Consider three invariants (that depend on the direction of the vev):

$$-\tfrac{1}{2}v^2 I_1 \equiv \widehat{v}^*_{\bar{a}} Y_{a\bar{b}} Z^{(1)}_{b\bar{d}} \widehat{v}_d \,,$$
$$\tfrac{1}{4}v^4 I_2 \equiv \widehat{v}^*_{\bar{b}} \widehat{v}^*_{\bar{c}} Y_{b\bar{e}} Y_{c\bar{f}} Z_{e\bar{a}f\bar{d}} \widehat{v}_a \widehat{v}_d \,,$$
$$\tfrac{1}{4}v^2 I_3 \equiv \widehat{v}^*_{\bar{b}} \widehat{v}^*_{\bar{c}} Z^{(1)}_{b\bar{e}} \left[\tfrac{1}{4}v^2 Z^{(1)}_{c\bar{f}} Z_{e\bar{a}f\bar{d}} + Y_{e\bar{d}} Z^{(1)}_{c\bar{a}}\right] \widehat{v}_a \widehat{v}_d \,.$$

The factors of Y can be eliminated using the scalar potential minimum conditions, resulting in:

$$I_1 \equiv \widehat{v}^*_{\bar{a}} \widehat{v}^*_{\bar{e}} Z_{a\bar{b}e\bar{f}} Z^{(1)}_{b\bar{d}} \widehat{v}_d \widehat{v}_f \,,$$
$$I_2 \equiv \widehat{v}^*_{\bar{b}} \widehat{v}^*_{\bar{c}} \widehat{v}^*_{\bar{g}} \widehat{v}^*_{\bar{p}} Z_{b\bar{e}g\bar{h}} Z_{c\bar{f}p\bar{r}} Z_{e\bar{a}f\bar{d}} \widehat{v}_a \widehat{v}_d \widehat{v}_h \widehat{v}_r \,,$$
$$I_3 \equiv \widehat{v}^*_{\bar{b}} \widehat{v}^*_{\bar{c}} Z^{(1)}_{b\bar{e}} \left[Z^{(1)}_{c\bar{f}} Z_{e\bar{a}f\bar{d}} - 2 Z^{(1)}_{c\bar{d}} Z_{e\bar{a}f\bar{g}} \widehat{v}_g \widehat{v}^*_{\bar{f}}\right] \widehat{v}_a \widehat{v}_d \,.$$

Evaluating the CP-odd invariants in the so-called Higgs basis, where $\widehat{v} = (1,0)$, we end up with:

$$\mathrm{Im}\ I_1 = \mathrm{Im}[Z_6 Z_7^*]\,, \qquad \mathrm{Im}\ I_2 = \mathrm{Im}[Z_5^* Z_6^2]\,,$$
$$\mathrm{Im}\ I_3 = \mathrm{Im}[Z_5^*(Z_6+Z_7)^2]\,,$$

where Z_i are the scalar potential coefficients in the Higgs basis.

Theorem 5: The necessary and sufficient conditions for a CP-invariant scalar Higgs potential and a CP-conserving Higgs vacuum are: $\mathrm{Im}\ I_1 = \mathrm{Im}\ I_2 = \mathrm{Im}\ I_3 = 0$.

Although there are at most two independent relative phases among I_1, I_2 and I_3, there are cases where two of the three invariants are real, so all three must be checked. That is,

$$\mathrm{Im}\ I_1 = \mathrm{Im}\ I_2 = 0 \quad \text{if} \quad Z_6 = 0\,,$$

$$\mathrm{Im}\ I_1 = \mathrm{Im}\ I_3 = 0 \quad \text{if} \quad Z_7 = -Z_6\,,$$

$$\mathrm{Im}\ I_2 = \mathrm{Im}\ I_3 = 0 \quad \text{if} \quad Z_5 = 0\,,$$

Note: additional CP-odd invariants involving the Higgs-fermion Yukawa matrices enter when the Higgs-fermion couplings are included.

Unfinished business

• What is the physical significance of the four explicit CP-violating invariants? Which (if any) could be measured in precision Higgs studies at the ILC?

• Express all Higgs couplings in terms of invariants in the most general CP-violating 2HDM (and explore the approach to the decoupling limit).

• The CP-odd invariants are induced at the one-loop level in the MSSM. Find relations among these invariants which are perhaps a remnant of the underlying supersymmetry.

• Extend the basis independent formalism to include Higgs couplings to other sectors of the theory. (Some work along these lines already exists—see *CP Violation* by G. Branco *et al.* and references contained therein.)