Symmetries in $\mathcal{2HDM}$, $\mathcal{CP}$ violation and heavy Higgs effects

I.F. Ginzburg
Sobolev Inst. of Mathematics, SB RAS, Novosibirsk,
M. Krawczyk
ITP, Warsaw University

Outline

- **Lagrangian of $\mathcal{2HDM}$**
Reparametrization invariance and rephasing invariance
- **Z_2 symmetry and its violation**
- **Explicit $\mathcal{CP}$ violation description**
- **Heavy Higgs bosons with decoupling and without it**
- **Natural parameters range**

The simplest extension of the $\mathcal{SM}$ —
a Two Higgs Doublet Model (2$\mathcal{HDM}$):

$\mathcal{L} = \mathcal{L}_{gf}^{SM} + \mathcal{L}_H + \mathcal{L}_Y$;

$\mathcal{L}_{gf}^{SM}$ – $\mathcal{SM}$ interaction, gauge bosons + fermions

$\mathcal{L}_H \equiv T - V$ – Higgs lagrangian,

T – Higgs kinetic term, V – Higgs potential,

$\mathcal{L}_Y$ – Yukawa interaction of fermions to scalars.

$$T = (D_\mu\phi_1)^\dagger(D^\mu\phi_1) + (D_\mu\phi_2)^\dagger(D^\mu\phi_2)$$

$$+ \boxed{\varkappa(D_\mu\phi_1)^\dagger(D^\mu\phi_2) + \varkappa^*(D_\mu\phi_2)^\dagger(D^\mu\phi_1)},$$

$$V = \frac{\lambda_1}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_2}{2}(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2)$$

$$+\lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \frac{1}{2}\left[\lambda_5(\phi_1^\dagger\phi_2)^2 + h.c.\right]$$

$$+\left\{\left[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right](\phi_1^\dagger\phi_2) + h.c.\right\} + \mathcal{M}(\phi_i)$$

$$\mathcal{M}(\phi_i) = -\frac{1}{2}\left\{m_{11}^2(\phi_1^\dagger\phi_1) + m_{22}^2(\phi_2^\dagger\phi_2)\right.$$

$$\left. + \left[m_{12}^2(\phi_1^\dagger\phi_2) + h.c.\right]\right\}.$$

λ_{5-7}, $\varkappa$ and m_{12} are generally complex.

Reparameterization invariance

Two fields with identical quantum numbers $\Rightarrow$ Model can be described both in terms of fields ϕ_i and in terms of fields ϕ'_i:

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \hat{\mathcal{F}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} ,$$

$$\hat{\mathcal{F}} = e^{-i\rho_0} \begin{pmatrix} \cos\theta\, e^{i\rho/2} & \sin\theta\, e^{i(\tau-\rho/2)} \\ -\sin\theta\, e^{-i(\tau-\rho/2)} & \cos\theta\, e^{-i\rho/2} \end{pmatrix} .$$

In the $\varkappa = 0$ case this transformation does not change the form of kinetic term but induce the changes of coefficients of Lagrangian, which we call a reparametrization transformation – RPT.

Denoting $c = \cos\theta$, $s = \sin\theta$, $\mu_{12}^2 = \text{Re}\,(m_{12}^2 e^{-i\tau})$, $\tilde\lambda_5 = \lambda_5 e^{-2i\tau}$, $\tilde\lambda_{6,7} = \lambda_{6,7} e^{-i\tau}$, the RPT is described by eq-s:

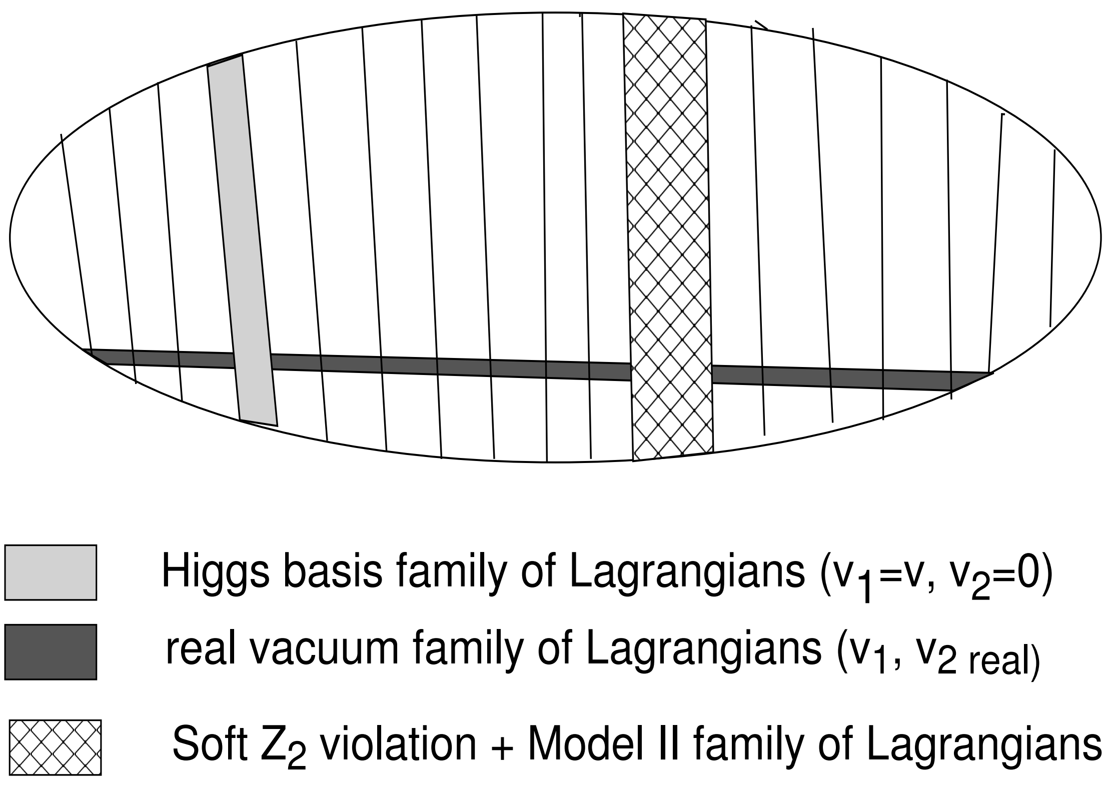
$$\lambda_1' = c^2\lambda_1 + s^2\lambda_2 - cs\Phi - 2cs\,\text{Re}\,(\tilde\lambda_6 + \tilde\lambda_7),$$
$$\lambda_2' = s^2\lambda_1 + c^2\lambda_2 - cs\Phi + 2cs\,\text{Re}\,(\tilde\lambda_6 + \tilde\lambda_7),$$
$$\lambda_3' = \lambda_3 + cs\Phi, \qquad \lambda_4' = \lambda_4 + cs\Phi,$$
$$e^{2i\rho}\lambda_5' = \lambda_5 + e^{i\tau}\left[cs\Phi + 2is^2\,\text{Im}\,\tilde\lambda_5 - 2ics\,\text{Im}\,(\tilde\lambda_6 - \tilde\lambda_7)\right],$$
$$\lambda_6' = e^{-i\rho}\left[c^2\lambda_6 - s^2\lambda_7 + \frac{e^{i\tau}}{2}cs(\lambda_1 - \lambda_2 + \Psi)\right],$$
$$\lambda_7' = e^{-i\rho}\left[c^2\lambda_7 - s^2\lambda_6 + \frac{e^{i\tau}}{2}cs(\lambda_1 - \lambda_2 - \Psi)\right],$$

$$(m')_{11}^2 = c^2 m_{11}^2 + s^2 m_{22}^2 - 2cs\mu_{12}^2,$$
$$(m')_{22}^2 = s^2 m_{11}^2 + c^2 m_{22}^2 + 2cs\mu_{12}^2,$$
$$(m')_{12}^2 = e^{-i\rho}\left\{m_{12}^2 + e^{i\tau}\left[cs(m_{11}^2 - m_{22}^2) - 2s^2\mu_{12}^2\right]\right\}.$$

$$\Phi_0 = \lambda_1 + \lambda_2 - 2(\lambda_3 + \lambda_4 + \text{Re}\,\tilde\lambda_5),$$
$$\Phi = cs\Phi_0 + 2(c^2 - s^2)\,\text{Re}\,(\tilde\lambda_6 - \tilde\lambda_7),$$
$$\Psi = (c^2 - s^2)\Phi_0 - 8cs\,\text{Re}\,(\tilde\lambda_6 - \tilde\lambda_7) + 2i\,\text{Im}\,\tilde\lambda_5.$$

A set of Higgs Lagrangians, obtained from each other by this transformations, forms the reparametrization equivalent space of Lagrangians (RPES) – a 3-dimensional subspace of the entire space of Lagrangians. The different Lagrangians within this RPES are physically equivalent. That is

reparametrization invariance (RPI)



Schematic presentation of RPES. Different strips represent families of Lagrangians with different explicit properties. A particular case where the soft Z_2 violating and Model II Lagrangians families coincide is shown.

Some parameters of theory which are treated often as physical are in fact reparametrization dependent. The most important example – a ratio of v.e.v.'s of scalar fields, $\tan\beta$.

E.g., under the RPT with $\rho = \xi$, $\tau = 0$

$$\beta \Rightarrow \beta + \theta \quad .$$

Particular case

at $\theta = 0$:

2. Rephasing invariance

under the global rephasing transformation

$$\phi_i \to e^{-i\rho_i}\phi_i, \quad (i = 1, 2),$$

$$\rho_0 = (\rho_1 + \rho_2)/2, \quad \rho = \rho_2 - \rho_1 (\equiv 2\rho'),$$

This transformation leads to a rephasing transformation of the parameters:

$$\lambda_{1-4} \to \lambda_{1-4}\,, \quad m_{ii}^2 \to m_{ii}^2\,, \quad m_{12}^2 \to m_{12}^2 e^{i\rho}$$

$$\lambda_5 \to \lambda_5\, e^{2i\rho}\,, \quad \lambda_{6,7} \to \lambda_{6,7}\, e^{i\rho}\,, \quad \varkappa \to \varkappa\, e^{i\rho}\,.$$

By construction, the Lagrangian with coefficients λ_i, m_{ij}^2 and with new coefficients describe the same physical reality. We call this property a rephasing invariance

This invariance is extended to the description of a whole system of scalars and fermions by adding of similar transformations for the phases of fermion fields and Yukawa couplings.

The Z_2 symmetry and its violations

The $2\mathcal{HDM}$ generally give a $\not{C}P$ at $\mathcal{EWSB}$.
In the most general form of $\mathcal{L}_Y$ large $\mathcal{FCNC}$ effects become possible.
Experiment: $\not{C}P$ and $\mathcal{FCNC}$ effects are weak.

$$\Downarrow$$

The natural construction of $2\mathcal{HDM}$ should start with the lagrangian having additional symmetry which forbids a $\not{C}P$ and $\mathcal{FCNC}$ effects.

$$\Downarrow$$

That is Z_2 symmetry under independent transformations for both fields

$$\phi_1 \to -\phi_1, \ \phi_2 \to \phi_2, \quad \phi_1 \to \phi_1, \ \phi_2 \to -\phi_2,$$

which forbids (ϕ_1, ϕ_2) mixing.
This symmetry can be weakly broken to open door for weak $\not{C}P$ and $\mathcal{FCNC}$ effects.

Z_2 conserving case: $m_{12} = \lambda_6 = \lambda_7 = \varkappa = 0$.
Soft violation of Z_2: dim. 2 operator with m_{12} (retained unmixed ϕ_i fields at small distances).
Hard violation of Z_2: + dim. 4 operators with λ_6, λ_7, $\varkappa$ – looks unnatural since (ϕ_1, ϕ_2) mixing retains at small distances.

The case of hidden soft Z_2 violation

Let: physical model can be described by Lagrangian $\mathcal{L}_s$ with exact or softly violated Z_2 symmetry. The general RPT converts $\mathcal{L}_s \Rightarrow \mathcal{L}_{hs}$ – $\mathcal{L}$ of hidden soft Z_2 violation (λ_6, $\lambda_7 \neq 0$, $\varkappa = 0$). We first apply the RPT $\mathcal{L}_s \Rightarrow \mathcal{L}_s^R$, making λ_5 real (still m_{12}^2 can be complex leaving open an opportunity for CP violation). Then a general RPT transforms $\mathcal{L}_s^R \Rightarrow \mathcal{L}_{hs}$ with quartic sector, which is described by 8 independent parameters (λ_{1-5} and θ, ρ, τ) instead of 10 independent parameters of the general Lagrangian (λ_{1-4}, $\mathrm{Re}\,\lambda_{5-7}$, $\mathrm{Im}\,\lambda_{5-7}$):

$$\lambda_1' = c^2\lambda_1 + s^2\lambda_2 - cs\Phi, \lambda_2' = s^2\lambda_1 + c^2\lambda_2 - cs\Phi,$$

$$\lambda_3' = \lambda_3 + cs\Phi, \qquad \lambda_4' = \lambda_4 + cs\Phi,$$

$$\lambda_5' = e^{-2i\rho}\lambda_5 + e^{2i\tau}[cs\Phi + 2is^2\lambda_5 \sin 2\tau],$$

$$\lambda_6' = \frac{e^{i(\tau-\rho)}}{2}\left[cs(\lambda_1 - \lambda_2) + A\right],$$

$$\lambda_7' = \frac{e^{i(\tau-\rho)}}{2}\left[cs(\lambda_1 - \lambda_2) - A\right],$$

with

$$A = (c^2 - s^2)\Phi + 2ics\lambda_5 \sin 2\tau,$$

$$\Phi = cs[\lambda_1 + \lambda_2 - 2(\lambda_3 + \lambda_4 + \lambda_5 \cos 2\tau)].$$

These eq-s allow to find parameters of the Lagrangian $\mathcal{L}_s^R$ via parameters of $\mathcal{L}_{hs}$:

1) $$\frac{\lambda_6' + \lambda_7'}{\lambda_6'^* + \lambda_7'^*} = e^{2i(\tau-\rho)} \Rightarrow \tau - \rho\,;$$

2) $$\frac{\lambda_6' + \lambda_7'}{\lambda_1' - \lambda_2'} = e^{i(\tau-\rho)}\frac{\tan 2\theta}{2} \Rightarrow \theta\,;$$

3) $$e^{-i(\tau-\rho)}(\lambda_6' - \lambda_7') = (c^2 - s^2)\Phi + 2ics\lambda_5 \sin 2\tau$$
$$\Rightarrow \Phi \; and \; 2cs\lambda_5 \sin 2\tau;$$

4) $$e^{-i\rho}\lambda_5 = \lambda_5' - e^{2i(\tau-\rho)}[cs\Phi + 2is^2 \sin 2\tau\lambda_5]$$
$$\Rightarrow \rho \; and \; \lambda_5\,;$$

5) Finally, all remaining quantities λ_{1-4} can be determined easily from the first four equations.

True hard violation of Z_2

1) The (ϕ_1, ϕ_2) mixing retains at small distances – very unnatural .

2) The mixed kinetic terms (with $\varkappa$, $\varkappa^*$) can be removed by the nonunitary transformation:

$$(\phi_1', \phi_2') \to \left(\frac{\sqrt{\varkappa^*}\phi_1 + \sqrt{\varkappa}\phi_2}{2\sqrt{|\varkappa|(1+|\varkappa|)}} \pm \frac{\sqrt{\varkappa^*}\phi_1 - \sqrt{\varkappa}\phi_2}{2\sqrt{|\varkappa|(1-|\varkappa|)}} \right) . \quad \blacklozenge$$

Starting from the case $\varkappa = 0$, $\lambda_{6,7} \neq 0$, the renormalization of quadratically divergent, non-diagonal two-point functions leads to $\varkappa \neq 0 \Rightarrow$ λ_6, λ_7, $\varkappa$ are running $\Rightarrow$ all of these terms should be included in Lagrangian on the same footing $\Rightarrow$ the treatment of the hard violation of Z_2 symmetry without $\varkappa$ terms (as in most of papers considering this "most general $2\mathcal{HDM}$ potential") is inconsistent.

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The diagonalization $\blacklozenge$ destroy relatively simple relations for the masses of the Higgs bosons, usually written.

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We present relations for a case of hard violation of Z_2 symmetry at $\varkappa = 0$ keeping in mind that the loop corrections can change results significantly.

The extremes of the potential

define the v.e.v.'s $\langle\phi_i\rangle$ via

$$\boxed{\frac{\partial V}{\partial\phi_i}(\phi_1=\langle\phi_1\rangle,\;\; \phi_2=\langle\phi_2\rangle)=0.}$$

With accuracy to the choice of z axis in the weak isospin space, most general solution has form

$$\langle\phi_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_1\end{pmatrix},\quad \langle\phi_2\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}u\\ v_2e^{i\xi}\end{pmatrix};$$

Denote

$y_1=\langle\phi_1^\dagger\rangle\langle\phi_1\rangle$, $y_2=\langle\phi_2^\dagger\rangle\langle\phi_2\rangle$, $y_3=\langle\phi_1^\dagger\rangle\langle\phi_2\rangle$.

There are 2 solutions of extremum condition

1) with $\boxed{y_3^*y_3-y_1y_2\neq 0}\Rightarrow\boxed{u\neq 0}$.

The v.e.v.'s are given by eq-s

$$\lambda_1y_1+\lambda_3y_2+\lambda_6^*y_3^*+\lambda_6y_3=m_{11}^2/2,$$
$$\lambda_2y_2+\lambda_3y_1+\lambda_7^*y_3^*+\lambda_7y_3=m_{22}^2/2,$$
$$\lambda_4y_3^*+\lambda_5y_3+\lambda_6y_1+\lambda_7y_2=m_{12}^2/2.$$

For some set of parameters of $\mathcal{L}$ this solution describe minimum of the potential

$\Rightarrow$ Charged vacuum, with massive photon **!**

It does not describe reality.

Standard vacuum

Another solution of extremum condition

2) with $y_3^* y_3 = y_1 y_2 \Rightarrow u = 0$.

$$\Rightarrow \langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_2 e^{i\xi} \end{pmatrix}.$$

Standard $v_1 = v\cos\beta$, $v_2 = v\sin\beta$ with the $\mathcal{SM}$ constraint $v = \left(G_F\sqrt{2}\right)^{-1/2} = 246$ GeV.

That is minimum of potential for those parameters of potential at which all eigenvalues of effective mass squared matrix in the extremum point are non-negative – at positive mass squared of all physical Higgs bosons (calculated below).

It can describe reality.

At this set of parameters the charged vacuum solution describe either saddle point or local (not global) minimum with larger vacuum energy (Diaz-Cruz et al).

The rephasing transformation shifts $\xi \to \xi - \rho$. Let us take some Lagrangian describing our model and calculate v.e.v.'s. Than, the rephasing transformation with $\rho = \xi$ gives a real vacuum form of Lagrangian -rvL with $\xi = 0$ (horizontal strip in figure) with

$$\lambda_{1-4,rv} = \lambda_{1-4},\ \ \lambda_{5,rv} = \lambda_5 e^{-2i\xi},$$
$$\lambda_{6,rv} = \lambda_6 e^{-i\xi},\ \ \lambda_{7,rv} = \lambda_7 e^{-i\xi},$$
$$\varkappa_{rv} = \varkappa e^{-i\xi},\ \ m_{12,rv}^2 = m_{12}^2 e^{-i\xi},$$

At given v_i we denote

$$\lambda_{345,rv} = \lambda_{3,rv} + \lambda_{4,rv} + \text{Re}\,\lambda_{5,rv},$$
$$\frac{v_1}{v_2}\lambda_{6,rv} \pm \frac{v_2}{v_1}\lambda_{7,rv} = \begin{cases} \lambda_{67,rv} & for\ +, \\ 2\tilde{\lambda}_{67,rv} & for\ -; \end{cases}$$
$$m_{12,rv}^2 = 2v_1 v_2(\nu + i\delta).$$

Beginning from here we use the rvL, without writing explicitly the subscript *r*v.

$$
\begin{aligned}
V = \frac{\lambda_1}{2}\left[(\phi_1^\dagger\phi_1) - \frac{v_1^2}{2}\right]^2 + \frac{\lambda_2}{2}\left[(\phi_2^\dagger\phi_2) - \frac{v_2^2}{2}\right]^2 \\
+\lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\
+\frac{1}{2}\left[\lambda_5(\phi_1^\dagger\phi_2)^2 + \text{h.c.}\right] \\
+\left\{\left[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right](\phi_1^\dagger\phi_2) + \text{h.c.}\right\} \\
-\frac{1}{2}\Big(\lambda_{345} + Re\lambda_{67}\Big)[v_2^2(\phi_1^\dagger\phi_1) + v_1^2(\phi_2^\dagger\phi_2)] \\
-v_1v_2\, Re[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)] \\
+\nu(v_2\phi_1 - v_1\phi_2)^\dagger(v_2\phi_1 - v_1\phi_2) \\
+2\delta\cdot v_1v_2 Im(\phi_1^\dagger\phi_2).
\end{aligned}
$$

Mass term here is written via v_1, v_2 and λ's

plus

single free dimensionless parameter ν.

The imaginary part of m_{12}^2 is constrained in rvL by relation

$$Im(m_{12}^2) \equiv 2v_1v_2\delta = Im(\lambda_5 + \lambda_{67})v_1v_2$$

The standard decomposition of the fields ϕ_i in terms of physical fields (in zero rephasing gauge):

$$\phi_i = \begin{pmatrix} \varphi_i^+ \\ \frac{1}{\sqrt{2}}(v_i + \eta_i + i\chi_i) \end{pmatrix} \quad (i = 1, 2).$$

Goldstone boson fields

$$G^0 = \cos\beta\,\chi_1 + \sin\beta\,\chi_2,$$
$$G^\pm = \cos\beta\,\varphi_1^\pm + \sin\beta\,\varphi_2^\pm.$$

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Charged Higgs boson fields

$$H^\pm = \sin\beta\,\varphi_1^\pm + \cos\beta\,\varphi_2^\pm \text{ with}$$
$$M_{H^\pm}^2 = v^2\left[\nu - \frac{1}{2}Re(\lambda_4 + \lambda_5 + \lambda_{67})\right].$$

Neutral Higgs sector. By definition η_i are standard $C-$ and $P-$ even (scalar) fields while $A = -\sin\beta\,\chi_1 + \cos\beta\,\chi_2$ is C–odd (in the interactions with fermions it behaves as $P-$ odd particle - a pseudoscalar). The mass-squared matrix $\mathcal{M}$ in the η_1, η_2, A basis is

$$\mathcal{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}, \qquad \text{with}$$

$$\begin{aligned}
M_{11} &= \left[c_\beta^2\,\lambda_1 + s_\beta^2\,\nu + s_\beta^2 Re(\lambda_{67}/2 + 2\tilde{\lambda}_{67})\right] v^2, \\
M_{22} &= \left[s_\beta^2\,\lambda_2 + c_\beta^2\,\nu + c_\beta^2 Re(\lambda_{67}/2 - 2\tilde{\lambda}_{67})\right] v^2, \\
M_{12} &= -\left(\nu - \lambda_{345} - \frac{3}{2} Re\lambda_{67}\right) c_\beta s_\beta v^2, \\
M_{13} &= -\left(\delta + Im\tilde{\lambda}_{67}\right) s_\beta v^2, \\
M_{23} &= -\left(\delta - Im\tilde{\lambda}_{67}\right) c_\beta v^2, \\
M_{33} &= \left[\nu - Re(\lambda_5 - \frac{1}{2}\lambda_{67})\right] v^2 \equiv M_A^2, \\
c_\beta &= \cos\beta, \qquad s_\beta = \sin\beta.
\end{aligned}$$

M_A is CP–odd Higgs boson mass in the CP conserving case.

The masses squared of the physical neutral states h_i – eigenvalues of the matrix $\mathcal{M}$, the Higgs eigenstates h_i have no definite $\mathcal{CP}$ parity since they are mixtures of fields η_i and A with opposite $\mathcal{CP}$ parities (provided by M_{13} and M_{23}):

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = R \begin{pmatrix} \eta_1 \\ \eta_2 \\ A \end{pmatrix} \quad \text{with } R\mathcal{M}R^T = diag(M_1^2, M_2^2, M_3^2)\,.$$

The diagonalizing matrix

$$R = R_3 R_2 R_1 \equiv \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$

$$R_1 = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix}.$$

(R_i are rotation matrices, α_i are Euler angles, $c_i = \cos\alpha_i$, $s_i = \sin\alpha_i$).

Two step diagonalization

1. Scalar (12) sector

$$\begin{pmatrix} h \\ -H \\ A \end{pmatrix} = R_1 \begin{pmatrix} \eta_1 \\ \eta_2 \\ A \end{pmatrix} \quad \text{with}$$

$$R_1 \mathcal{M} R_1^T = \mathcal{M}_1 \equiv \begin{pmatrix} M_h^2 & 0 & M'_{13} \\ 0 & M_H^2 & M'_{23} \\ M'_{13} & M'_{23} & M_A^2 \end{pmatrix},$$

$$M'_{13} = c_1 M_{13} + s_1 M_{23}, \quad M'_{23} = -s_1 M_{13} + c_1 M_{23}.$$

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If $\mathcal{CP}$ conserves (at $M_{13} = M_{23} = 0$), $h_1 = h$, $h_2 = -H$, $h_3 = A$. So, notations customary for $\mathcal{CP}$ conserving case:

$$\boxed{\alpha = \alpha_1 - \pi/2}, \quad \alpha \in (-\pi/2, \pi/2)\,.$$

$$H = \cos\alpha\, \eta_1 + \sin\alpha\, \eta_2\,, \quad h = -\sin\alpha\, \eta_1 + \cos\alpha\, \eta_2\,,$$

$$M_{h,H}^2 = (M_{11} + M_{22} \mp \mathcal{N})\,/2,$$

$$\mathcal{N} = \sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2}\,,$$

$$\sin 2\alpha = \frac{2M_{12}}{M_H^2 - M_h^2} \Rightarrow \frac{\sin 2\alpha}{\sin 2\beta} = \frac{v^2(\lambda_{345} - \nu)}{M_H^2 - M_h^2},$$

$$M'_{13} = -v^2[\delta \cos(\beta + \alpha) - Im\tilde{\lambda}_{67} \cos(\beta - \alpha)]\,,$$

$$M'_{23} = v^2[\delta \sin(\beta + \alpha) - Im\tilde{\lambda}_{67} \sin(\beta - \alpha)]\,.$$

2. Complete diagonalization

The above diagonalization keeps two off-diagonal elements in mass matrix $\mathcal{M}_1$, which are combined from $\delta(\propto \text{Im}\,(m_{12}^2))$ and $Im\tilde{\lambda}_{67}$. If at least one of these terms $\neq 0$, the additional diagonalization is necessary, and the mass eigenstates, being admixtures of CP–even and CP–odd states, violate CP symmetry.

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = R_3 R_2 \begin{pmatrix} h \\ -H \\ A \end{pmatrix} \quad \text{with}$$

$$R\,\mathcal{M}\,R^T = R_3 R_2 \mathcal{M}_1 R_2^T R_3^T = \begin{pmatrix} M_1^2 & & \\ & M_2^2 & \\ & & M_3^2 \end{pmatrix}$$

The angles α_2 and α_3 describe mixing of $\mathcal{CP}$ – even states h, H with $\mathcal{CP}$ –odd state A.

$\Rightarrow$ The complexity of some parameters of the potential in its real vacuum form is necessary and sufficient condition for CP violation in the Higgs sector.

For an arbitrary form of Lagrangian (i.e. not for the real vacuum form) the necessary condition for CP violation in the Higgs sector can be written as complexity of at least one rephasing invariant combination

$$\lambda_5^*(m_{12}^2)^2\,, \quad \lambda_6^* m_{12}^2\,, \quad \lambda_7^* m_{12}^2\,.$$

Natural set of parameters

To have $\not{C}\!P$ in the Higgs sector $\Leftarrow Im\,(m_{12rv}^2) \neq 0$ (simultaneously $Im\,(\lambda_{5rv}) \neq 0$). This $\not{C}\!P$ is presumably weak if

$$Im\,(m_{12rv}^2) \ll |M_A^2 - M_h^2|,\, |M_A^2 - M_H^2|\,.$$

This simple form of condition is valid only for rvL. In other rephasing forms this condition includes both $Im\,(m_{12}^2)$ and $Re\,(m_{12}^2)$.

Naturally, this condition must be formulated independently on the rephasing gauge $\Rightarrow$ for the <u>natural set of parameters</u> of $2\mathcal{HDM}$ we require that $|m_{12}^2| \ll |M_A^2 - M_h^2|,\, |M_A^2 - M_H^2|$, i.e. $|\nu|,\ |\lambda_5| \ll |\lambda_{1-4}|$ *(natural set of parameters).*

Weak $\not{C}\!P$ in Higgs sector looks unnatural if $|\boldsymbol{m_{12}}|$ is large, i.e. a weak $\boldsymbol{\mathcal{CP}}$ violation naturally correspond to weakly broken $\boldsymbol{Z_2}$ symmetry with $|\boldsymbol{\nu}| < |\boldsymbol{\lambda_i}|$.

Special cases

• If $\delta = 0$ and $Im\tilde{\lambda}_{67} = 0$, CP symmetry does not violated, h, H and A are physical Higgs bosons and $\alpha_2 = \alpha_3 = 0$.

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• If $|M'_{13}/(M_A^2 - M_h^2)| \ll 1 \Rightarrow$
$\alpha_2 \approx 0 \Rightarrow h_1 \approx h$ (*practically* $\mathcal{CP}$ *–even*),
h_2, h_3 *generally have no definite* $\mathcal{CP}$ *parity*

$$\tan 2\alpha_3 \approx \frac{2M'_{23}}{M_A^2 - M_H^2}.$$

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• If $|M'_{23}/(M_A^2 - M_H^2)| \ll 1 \Rightarrow$
$\alpha_3 \approx 0 \Rightarrow h_2 \approx -H$ (*practically* $\mathcal{CP}$ *–even*),
h_1, h_3 *generally have no definite* $\mathcal{CP}$ *parity*

$$\tan 2\alpha_2 \approx \frac{2M'_{13}}{M_A^2 - M_h^2}.$$

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• Case of weak $\mathcal{CP}$ violation joins 2 above cases.

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• Intensive coupling regime $M_h \approx M_H \approx M_A$.
$\Rightarrow$ CP violating mixing of fields is naturally strong, spacing between M_i is increased due to this mixing.

Relative couplings of Higgs boson h_i:

$$\boxed{\chi_a^i \stackrel{def}{=} g_a^i/g_a^{SM}}, \qquad a = q,\, \ell,\, V(= Z,\, W)$$

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Couplings to gauge bosons

$$\chi_V^{(i)} = \cos\beta\, R_{i1} + \sin\beta\, R_{i2}, \quad i = 1-3, \quad V = W,\, Z$$

In particular, for the case with weak violation of $\mathcal{CP}$ symmetry approximately

$$\chi_V^{(1)} = \sin(\beta-\alpha), \quad \chi_V^{(2)} = -\cos(\beta-\alpha),$$
$$\chi_V^{(3)} = -s_2\sin(\beta-\alpha) + s_3\cos(\beta-\alpha).$$

Yukawa interaction

General Yukawa Lagrangian

$$-\mathcal{L}_{\mathsf{Y}} = \bar{Q}_L[(\Gamma_1\phi_1 + \Gamma_2\phi_2)d_R$$
$$+(\Delta_1\tilde{\phi}_1 + \Delta_2\tilde{\phi}_2)u_R] + \text{h.c.}$$
$$+ \textit{lepton terms}$$

Γ and Δ — 3–dimensional in the family space matrices with generally complex coefficients.

If they are non diagonal in family index, the $\mathcal{FCNC}$ appears.

> To have only soft violation of Z_2 symmetry (to keep separate fields ϕ_i at small distances), each right-handed fermion should couple to only one field, either ϕ_1 or ϕ_2.

Otherwise, e.g. in Model III, hard violation of Z_2 symmetry appears via one–loop corrections.

The case $\Gamma_2 = \Delta_2 = 0$ – Model I,

the case $\Gamma_2 = \Delta_1 = 0$ – Model II.

Model II

$$-\mathcal{L}_Y^{II} = \sum_{k=1,2,3} g_{dk}\bar{Q}_{Lk}\phi_1 d_{Rk} + \sum_{k=1,2,3} g_{uk}\bar{Q}_{Lk}\tilde{\phi}_2 u_{Rk}$$
$$+ \sum_{k=1,2,3} g_{\ell k}\bar{\ell}_{Lk}\phi_1 \ell_{Rk} + \text{h.c.}$$

For the physical Higgs fields it result in (for two-component spinors)

$$\chi_u^{(i)} = \frac{1}{\sin\beta}[R_{i2} - i\cos\beta\, R_{i3}],$$
$$\chi_d^{(i)} = \frac{1}{\cos\beta}[R_{i1} - i\sin\beta\, R_{i3}].$$

For the interaction of the charged Higgs bosons with fermions, independent on details of the Higgs potential, one has for 4-component spinors

$$\mathcal{L}_{H^- tb} = \frac{M_t}{v\sqrt{2}}\cot\beta\, \bar{b}(1+\gamma^5)H^- t$$
$$+\frac{M_b}{v\sqrt{2}}\tan\beta\, \bar{b}(1-\gamma^5)H^- t + h.c.$$

Useful relations

The unitarity of the mixing matrix R allows to obtain a number of relations between the relative couplings of neutral Higgs particles to the gauge bosons and fermions.

Reparam. invariant relations

• The pattern relation among the basic relative couplings of each neutral Higgs particle h_i (GKO):

$$(\chi_u^{(i)} + \chi_d^{(i)})\chi_V^{(i)} = 1 + \chi_u^{(i)}\chi_d^{(i)}, \qquad (pr)$$

• A vertical sum rule for each basic relative coupling χ_j to all three neutral Higgs bosons h_i (Gunion et al):

$$\sum_{i=1}^{3} (\chi_j^{(i)})^2 = 1 \qquad (j = V, d, u). \qquad (vsr)$$

• The relations for CP violated parts of Yukawa:

$$(1 - |\chi_d^{(i)}|^2)\ \mathrm{Im}\,\chi_u^{(i)} + (1 - |\chi_u^{(i)}|^2)\ \mathrm{Im}\,\chi_d^{(i)} = 0\,.$$

Reparam. non-invariant relations

are valid for the Model II form of Lagrangian.

- A horizontal sum rule for each neutral Higgs boson h_i (Gunion et al)

$$|\chi_u^{(i)}|^2 \sin^2\beta + |\chi_d^{(i)}|^2 \cos^2\beta = 1. \quad (hsr)$$

- Linear relation

$$\chi_V^{(i)} = \cos^2\beta\, \chi_d^{(i)*} + \sin^2\beta\, \chi_u^{(i)} =$$
$$= \cos^2\beta\, \chi_d^{(i)} + \sin^2\beta\, \chi_u^{(i)*}$$

- Besides,

$$\tan^2\beta = \frac{(\chi_V^{(i)} - \chi_d^{(i)})^*}{\chi_u^{(i)} - \chi_V^{(i)}} = \frac{\mathrm{Im}\,\chi_d^{(i)}}{\mathrm{Im}\,\chi_u^{(i)}} = \frac{1 - |\chi_d^{(i)}|^2}{|\chi_u^{(i)}|^2 - 1}.$$

The consequences for some cases
with possible CP violation everywhere

(i) $\chi_V^{(2)} \approx \pm 1 \Rightarrow \chi_V^{(1)} \approx \chi_V^{(3)} \approx 0$ independently on the form of Yukawa sector $\Leftarrow$ vsr.

(ii) $\chi_V^{(2)} \approx \pm 1 \Rightarrow (1 \mp \chi_d^{(2)})(1 \mp \chi_d^{(2)}) \approx 0 \Leftarrow$ pr.

(iii) $\chi_V^{(2)} \approx \pm 1 \Rightarrow \chi_u^{(1)}\chi_d^{(1)},\ \chi_u^{(3)}\chi_d^{(3)} \approx -1 \Leftarrow$ pr, vsr.

(iv) The couplings to fermions are generally complex $\chi_{u,d}^{(2)} \approx \pm 1 \Rightarrow \chi_{u,d}^{(1)} \approx \pm(\mp) i \chi_{u,d}^{(3)} \Leftarrow$ vsr.

(v) $\chi_u^{(i)} \approx \pm 1 \Rightarrow \chi_d^{(i)} \approx \pm(\mp)1 \Leftarrow$ hsr.

(vi) $|\chi_{u,d}^{(i)}| \gg 1 \Rightarrow \chi_{d,u}^{(i)} \approx 0 \Leftarrow$ hsr.

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In the $\mathcal{CP}$ conserving case

$$\chi_{H^\pm}^{(\phi)} \equiv -\frac{v g_{hH^+H^-}}{2M_{H^\pm}^2}$$

$$= \left(1 - \frac{M_\phi^2}{2M_{H^\pm}^2}\right)\chi_V^{(\phi)} + \frac{M_\phi^2 - \nu v^2}{2M_{H^\pm}^2}(\chi_u^{(\phi)} + \chi_d^{(\phi)}).$$

Constraints for parameters of Higgs potential

were written only in the case of soft violation of Z_2 symmetry without $\mathcal{CP}$ violation. We extend these results to the case with $\mathcal{CP}$ violation.

• Positivity (vacuum stability) constraints.

The potential must be positive at large quasi–classical values of fields $|\phi_i|$ for an arbitrary direction in the (ϕ_1, ϕ_2) plane:

$$\lambda_1 > 0, \;\; \lambda_2 > 0, \;\; \lambda_3 + \sqrt{\lambda_1 \lambda_2} > 0,$$
$$\lambda_3 + \lambda_4 - |\lambda_5| + \sqrt{\lambda_1 \lambda_2} > 0.$$

• Minimum constraints — conditions ensuring that the condition for vacuum is a local minimum for all directions in (ϕ_1, ϕ_2) space, except the Goldstone modes (the physical fields provide the basis in the coset).

• Unitarity constraints. The quartic terms of Higgs potential lead, in the tree approximation, to a s–wave Higgs-Higgs and $W_L W_L$ and $W_L H$, etc. scattering amplitudes for different elastic channels. These amplitudes should not overcome unitary limit for partial wave. The earlier constraints for the case without $\mathcal{CP}$ violation (Akeroyd et al.) − with real λ_5 extends to the case with $\mathcal{CP}$ violation by the change $\lambda_5 \to |\lambda_5|$(IFG, Ivanov).

These constraints give bounds for the Higgs-boson masses which strongly depend on the quadratic mass parameter ν.

Large ν $\Rightarrow$ all M_H, M_A, $M_{H^\pm}$ are large (decoupling limit).

Small ν $\Rightarrow$ moderately large upper bound of $600 \div 700$ GeV for M_H, M_A, $M_{H^\pm}$.

The correspondence between the tree-level unitarity limit and realization of the Higgs field as more or less narrow particle, as in minimal $\mathcal{SM}$, takes place in the $2\mathcal{HDM}$ only in the case when all unitarity constraints are violated simultaneously. In the case when only some of these constraints are violated the physical picture become more complex.

Heavy Higgs bosons in $2\mathcal{HDM}$

Many analyses of $2\mathcal{HDM}$ assume that the lightest Higgs boson h_1 is similar to the Higgs boson of the $\mathcal{SM}$, all other Higgs bosons are very heavy (with mass $\sim M$).

Usual additional hidden requirement (?!?):

Theory must have explicit decoupling property: the mention features remain valid at $M \to \infty$ (decoupling property).

In fact, the mentioned physical picture can be realized in the $2\mathcal{HDM}$ both with and without decoupling property.

Two scenarios of generation of heavy Higgs masses.

Decoupling of heavy Higgs bosons

is realized at unnatural condition $\nu \gg \|\lambda_i|$,

$\Rightarrow M'_{13} \sim \lambda_i v^2 \;\Rightarrow |M'_{13}| \ll M_A^2 - M_h^2 \approx \nu v^2 \Rightarrow$ $h_1 \approx h$, etc. as it was discussed earlier, $\beta - \alpha \approx \pi/2$,

$$M_h^2 = v^2 \left(\underbrace{c_\beta^4 \lambda_1 + s_\beta^4 \lambda_2 + 2 s_\beta^2 c_\beta^2 \lambda_{345}}_{soft} - \underbrace{2 s_\beta^2 c_\beta^2 Re\lambda_{67}}_{hard} \right)$$

$$M_H^2 = v^2 \left\{ \underbrace{\nu + s_\beta^2 c_\beta^2 (\lambda_1 + \lambda_2 - 2\lambda_{345})}_{soft} + \underbrace{Re\left[2 s_\beta c_\beta (\lambda_6 + \lambda_7) + \left(-\frac{3}{2} + 4 s_\beta^2 c_\beta^2 \right) \lambda_{67} \right]}_{hard} \right\},$$

$$\alpha \equiv \alpha_1 - \frac{\pi}{2} = \beta - \frac{\pi}{2} + \delta_\alpha\,,$$

$$\delta_\alpha = -\frac{\sin 2\beta [\lambda_{345} \cos 2\beta + c_\beta^2 \lambda_1 - s_\beta^2 \lambda_2 + \mathcal{O}(Re\lambda_{6,7})]}{\nu}$$

Decoupling. Lightest Higgs boson h_1.

$\beta - \alpha \approx \pi/2 \Rightarrow$ all couplings of h_1 are close to those in $\mathcal{SM}$ and also selfcouplings, $h_1h_1h_1$ and $h_1h_1h_1h_1$, are very close to the corresponding $\mathcal{SM}$ couplings. Besides, h_1 practically decouple from $H^\pm$, since the quantity $\chi^{(1)}_{H^\pm} \sim \mathcal{O}(|\lambda_i|/\nu)$.

Higgs bosons h_2, h_3 are almost degenerate in masses, since

$$M_A \approx M_H (\approx M_2 \approx M_3) = v\sqrt{\nu}\,(1 + \mathcal{O}\,(|\lambda|/\nu))\,.$$

Besides, $M_{H^\pm} \approx M_2 \approx M_3$.

The $\mathcal{CP}$ violating mixing angle α_3 can be large, $\tan 2\alpha_3 \approx \dfrac{2M'_{23}}{M_A^2 - M_H^2}$, and

$$\chi_u^{(2)} = i\chi_u^{(3)} = -\cot\beta\, e^{i\alpha_3},$$
$$\chi_d^{(2)} = i\chi_d^{(3)} = \tan\beta\, e^{-i\alpha_3}.$$

while couplings of h_2, h_3 to gauge bosons and $H^\pm$ are small,

$$\chi_V^{(2)} = \cos\alpha_3\delta_\alpha\,, \quad \chi_V^{(3)} = \sin\alpha_3\delta_\alpha\,,$$
$$\chi_{H^\pm}^{(2,3)} \sim \mathcal{O}(|\lambda_i|/\nu)\,.$$

Heavy Higgs bosons without decoupling.

The option, when except one neutral h_1 all other Higgs bosons are heavy enough, can also be realized in $2\mathcal{HDM}$ without decoupling (at natural set of parameters) .

Sets of parameters of potential, satisfying unitarity constraints, for light h (mass 120 GeV) and heavy H, $H^{\pm}$, non-decoupling case.

	$\tan\beta$	λ_1	λ_2	λ_3	λ_4	λ_5	ν
(1)	50	1	6	5.5	-6	-6	0.24
(2)	0.02	6	1	5.5	-6	-6	0.24
(3)	1	6.25	6.25	6.25	-6	-6	0
(4)	10	4	8	4.4	-9	-0.5 $+0.3i$	0.24

	M_h	M_H	M_A	$M_{H^\pm}$	s_2	s_3
(1)	120	600	600	600	-	-
(2)	120	600	600	600	-	-
(3)	120	600	600	600	-	-
(4)	120	700	206	556	0.09	0.02

Lines (1-3) – the case without $\mathcal{CP}$ violation, line (4) – with $\mathcal{CP}$ violation.