Bayesian Limits: The CDF7117 Study

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The plan for this talk is:

- Present a condensation of CDF7117, an investigation of the Bayesian approach to setting cross section upper limits.
- Mention available Bayesian limit calculating software.
- Concluding observations.

The problem: n events are observed in an experiment with Poisson probability

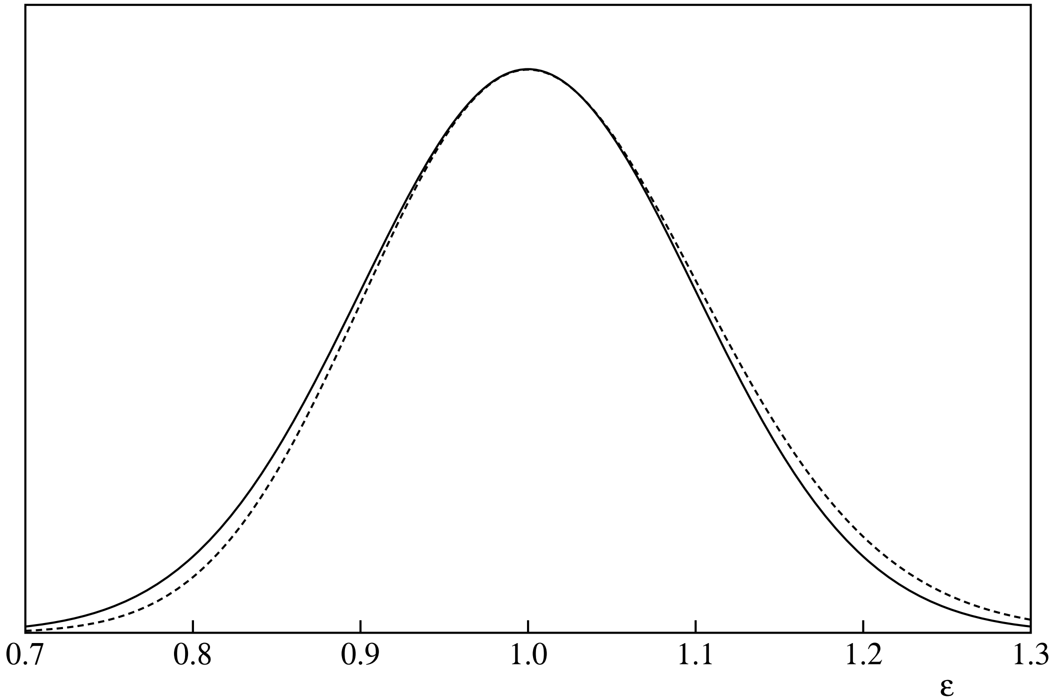
$$P(n) = \frac{e^{-(\epsilon s+b)}(\epsilon s + b)^n}{n!}$$

We wish to set an upper limit on the cross section s. The acceptance ϵ and the expected background b are known with some uncertainty. For simplicity, CDF7117 deals primarily with the case $\sigma_b = 0$; the extension to $\sigma_b > 0$ is straightforward.

The Bayesian approach then requires priors for s and ϵ (and for b, when $\sigma_b > 0$).

The Acceptance Prior

The acceptance (here, the product of efficiency and luminosity) is normally measured in a separate, subsidiary, measurement or calculation. For CDF7117, we defined a specific subsidiary measurement precisely, which led to a gamma distribution prior for the acceptance. For $\sigma_\epsilon = 10\%$, the gamma and Gaussian are quite similar:



acceptance prior p.d.f.

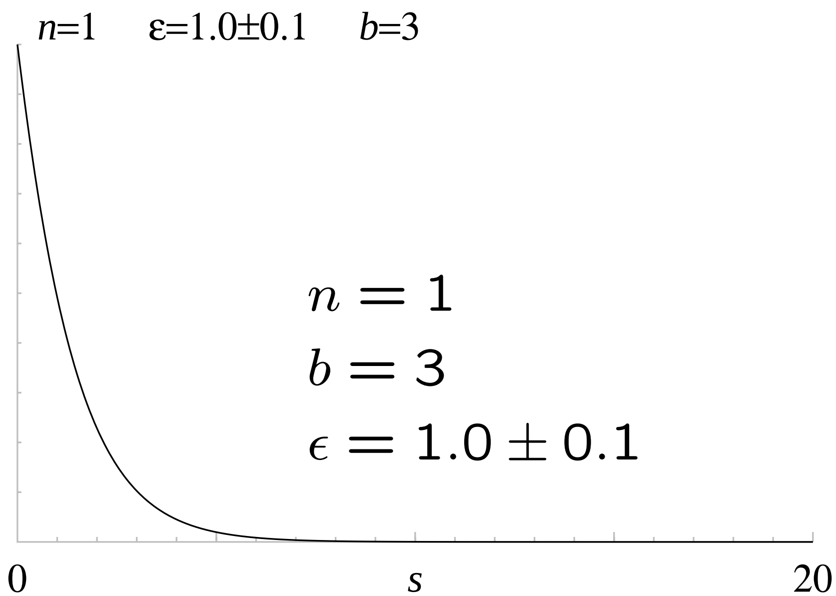
gamma: dashed
Gaussian: solid

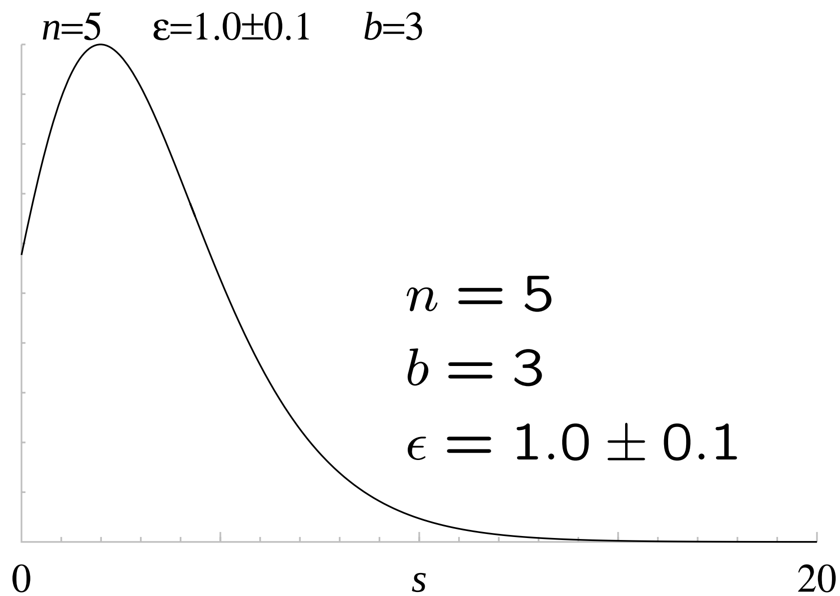
The Cross Section Prior

We take the prior for s to be flat for $s \geq 0$. This is well behaved when combined with a gamma prior for ϵ, but badly behaved with a truncated Gaussian ϵ prior. Luc Demortier will discuss a promising solution to this problem that involves a different cross section prior (the usual "cutoff" solution is unpleasant).

Marginalized Posterior p.d.f. for s

From the likelihood and the priors, we obtain the joint posterior p.d.f. for the cross section and nuisance parameters. Integrating over the nuisance parameters then yields the marginalized posterior p.d.f. for the cross section.





The posterior is the "answer" in the Bayesian approach; one should look at its shape before proceeding. One obtains an upper limit at credibility level β by finding s_{u} such that fraction β of the posterior is $< s_{\mathrm{u}}$.

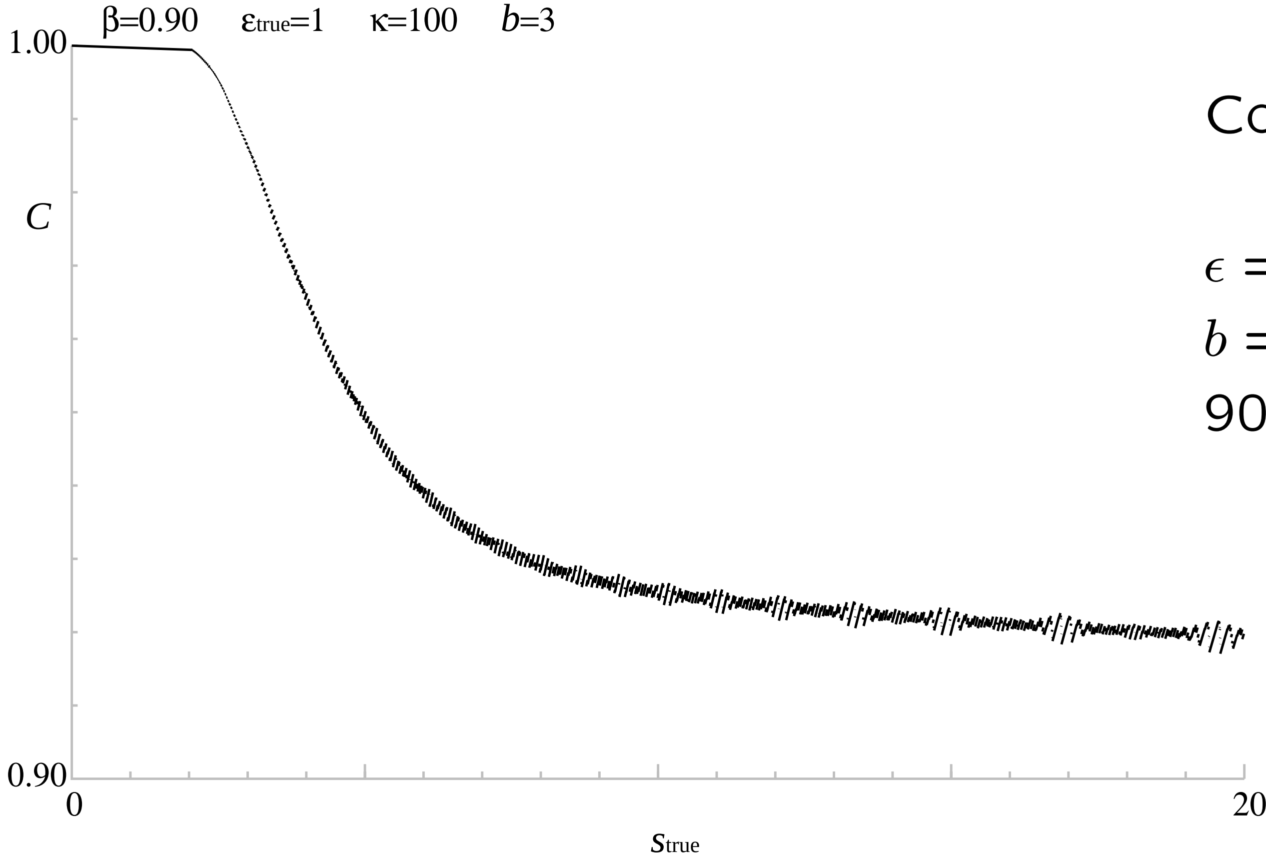
In the Bayesian approach, when $n = 0$, the posterior for s, and hence the limit, is completely independent of the expected background b (and σ_b). Conceptually, when $n = 0$, we know that there were exactly zero signal events and exactly zero background events, yielding effectively perfect separation of signal from background in this special case. (The same conclusion does not necessarily hold in a frequentist approach.)

90% Upper Limits

	$\epsilon = 1.0 \pm 0.1$							
n	$b = 0$	1	2	3	4	5	6	7
0	2.3531	2.3531	2.3531	2.3531	2.3531	2.3531	2.3531	2.3531
1	3.9868	3.3470	3.0620	2.9019	2.8000	2.7297	2.6783	2.6391
2	5.4669	4.5520	3.9676	3.6026	3.3623	3.1953	3.0736	2.9816
3	6.8745	5.8618	5.0463	4.4644	4.0571	3.7666	3.5534	3.3922
4	8.2380	7.1964	6.2451	5.4751	4.8914	4.4569	4.1313	3.8832
5	9.5714	8.5213	7.5063	6.6022	5.8579	5.2719	4.8180	4.4660
6	10.8826	9.8288	8.7885	7.8047	6.9344	6.2066	5.6184	5.1499
7	12.1766	11.1203	10.0703	9.0460	8.0904	7.2450	6.5289	5.9387
8	13.4570	12.3984	11.3441	10.3014	9.2952	8.3635	7.5374	6.8300
9	14.7261	13.6655	12.6085	11.5575	10.5247	9.5365	8.6250	7.8142
10	15.9858	14.9233	13.8641	12.8090	11.7630	10.7415	9.7701	8.8758
11	17.2375	16.1732	15.1121	14.0542	13.0017	11.9621	10.9525	9.9966
12	18.4823	17.4163	16.3533	15.2934	14.2371	13.1881	12.1560	11.1582
13	19.7210	18.6535	17.5887	16.5269	15.4682	14.4139	13.3692	12.3452
14	20.9545	19.8854	18.8191	17.7554	16.6946	15.6373	14.5856	13.5459
15	22.1832	21.1127	20.0448	18.9795	17.9169	16.8572	15.8014	14.7528
16	23.4078	22.3359	21.2665	20.1996	19.1353	18.0737	17.0151	15.9612
17	24.6286	23.5553	22.4845	21.4161	20.3502	19.2868	18.2261	17.1689
18	25.8459	24.7714	23.6992	22.6294	21.5619	20.4969	19.4344	18.3747
19	27.0601	25.9844	24.9109	23.8397	22.7708	21.7042	20.6400	19.5784
20	28.2715	27.1946	26.1199	25.0474	23.9770	22.9090	21.8432	20.7799

Coverage

The coverage probability, a function of the true value of the cross section (and nuisance parameters), is the (frequentist) probability that a repetition of the experiment will yield a limit that includes the true value of the cross section.



Coverage vs s_{true}

$\epsilon = 1.0 \pm 0.1$

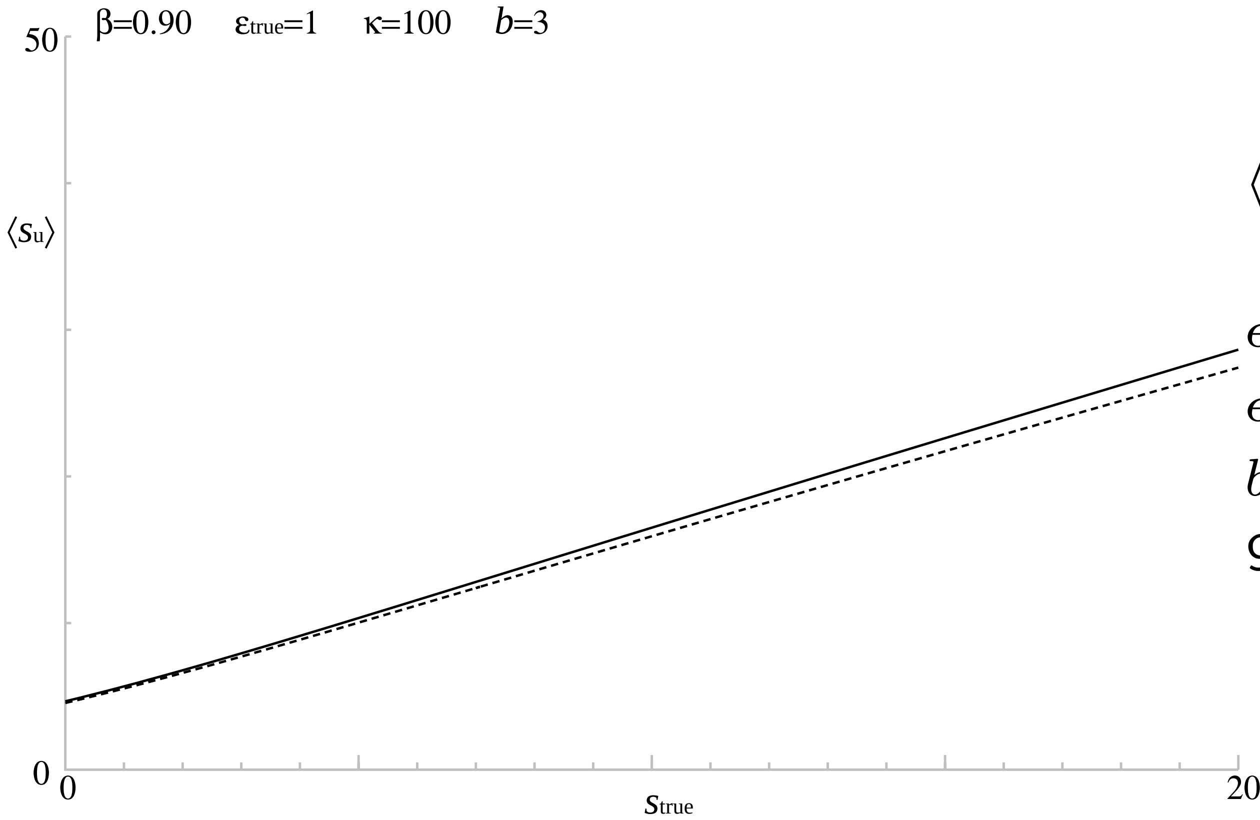
$b = 3$

90% upper limits

Bayesian methods achieve average coverage, as defined in CDF7117. Frequentists often want minimum coverage for this Poisson case, in which constant coverage is not possible. With an (improper) flat cross section prior, it appears that both average coverage and minimum coverage are achieved for upper limits in the Bayesian approach.

(We use credibility level in a Bayesian context, and confidence level in a frequentist context.)

Another quantity of interest is the expected limit, or sensitivity, as a function of the true cross section.



$\langle s_\mathrm{u} \rangle$ vs s_true

$\epsilon = 1 \pm 0.1$ solid
$\epsilon = 1 \pm 0$ dashed
$b = 3$
90% upper limits

Currently Available Bayesian Limit Software from CDF

`www-cdf.fnal.gov/physics/statistics/statistics_software.html`

- C code from CDF7117 study: gamma acceptance and background priors, various cross section priors. User can obtain posterior p.d.f., integral of posterior, upper limit at requested credibility level. Very fast execution time. User guide is CDF7232.

- `bayes.f` by John Conway. Traditional approach with truncated Gaussian priors for acceptance and background, flat prior with cutoff for cross section. Returns upper limit at requested credibility level. Slower because of MC integration. See CDF4476 by John Conway and Kaori Maeshima.

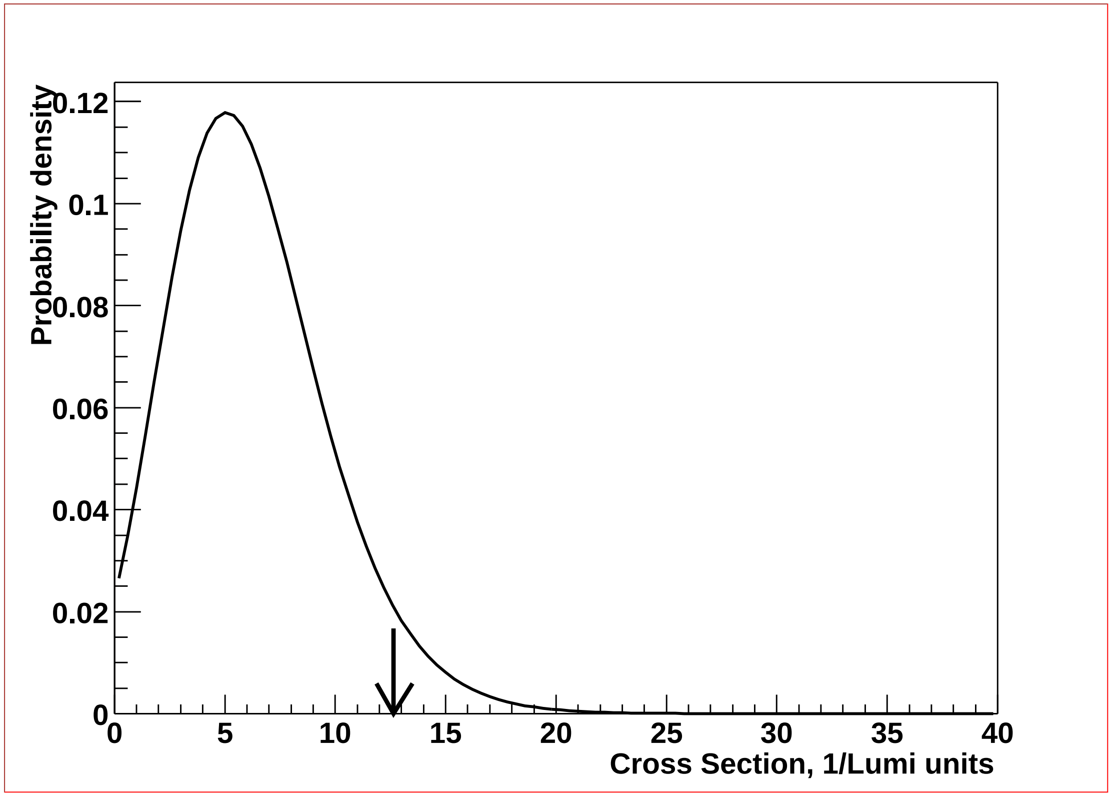
Other code available privately from John Conway:

```
bcorr.f   - multi-channel counting experiment with
            correlated and uncorrelated uncertainties
            (between channels, and between signal and
            background) a la CDF 6428

fit.f     - likelihood fit to spectrum with signal plus
            n background sources, including uncorrelated
            and correlated uncertainties on signal,
            background (implemented as nuisance parameters
            in the fit) described in CDF 6888
```

(`fit.f` is a profile likelihood approach.)

DØ (www-d0.fnal.gov/~hobbs/limit_calc.html) has a web based menu driven product which is interesting—similar to **bayes.f**, but produces limit (95% only) and posterior plot online:




```
Data:  10
Background:  5 +- 1
Efficiency:  1.0 +- 0.1
Luminosity:  1.0 +- 0.0
```

```
The cross section 95% CL upper limit is 12.666
```

Concluding Observations

- With a flat cross section prior and a gamma prior for acceptance, the Bayesian upper limit scheme exhibits no known defects. (Common defects are listed in CDF7117.)
- CDF7117 carries through all calculations analytically, which results in very fast computations. (Convenient for extensive coverage/sensitivity calculations.)
- Software is available for several combinations of priors.
- The gamma prior is more realistic than Gaussian for larger uncertainties (for small uncertainties they are very similar).
- Luc Demortier's correlated prior approach, described in the next talk, improves robustness, and represents a trivial modification to the existing code.