The chiral prediction for $a_0^0 - a_0^2$

Gilberto Colangelo

UNIVERSITÄT BERN

CERN, May 2, 2005

Outline

Low energy theorems, chiral expansion

Outline

Low energy theorems, chiral expansion

Dispersive methods

- Roy equations
- Chiral symmetry + dispersive methods

Outline

Low energy theorems, chiral expansion

Dispersive methods
- Roy equations
- Chiral symmetry + dispersive methods

What do we learn?

Low-energy theorem for $\pi\pi$ scattering

Some notation

$$\langle \pi^i \pi^j \text{ out} | \pi^k \pi^l \text{ in} \rangle = \delta^{ij}\delta^{kl} A(s,t,u) + \delta^{ik}\delta^{jl} A(t,u,s) + \delta^{il}\delta^{jk} A(u,t,s)$$

Low-energy theorem for $\pi\pi$ scattering

Some notation

$$\langle \pi^i \pi^j \text{ out} | \pi^k \pi^l \text{ in} \rangle = \delta^{ij}\delta^{kl} A(s,t,u) + \delta^{ik}\delta^{jl} A(t,u,s) + \delta^{il}\delta^{jk} A(u,t,s)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0}(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s)$$

Low-energy theorem for $\pi\pi$ scattering

Some notation

$$\langle\pi^i\pi^j \text{ out}|\pi^k\pi^l \text{ in}\rangle = \delta^{ij}\delta^{kl}A(s,t,u)+\delta^{ik}\delta^{jl}A(t,u,s)+\delta^{il}\delta^{jk}A(u,t,s)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0}(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s)$$

Low energy theorem

Weinberg 1966

$$A(s,t,u) = \frac{s-M_\pi^2}{F_\pi^2} + \mathcal{O}(p^4) \qquad \Rightarrow \qquad T^{I=0} = \frac{2s-M_\pi^2}{F_\pi^2}$$

Low-energy theorem for $\pi\pi$ scattering

Some notation

$$\langle \pi^i \pi^j \text{ out} | \pi^k \pi^l \text{ in} \rangle = \delta^{ij}\delta^{kl} A(s,t,u) + \delta^{ik}\delta^{jl} A(t,u,s) + \delta^{il}\delta^{jk} A(u,t,s)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0}(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s)$$

Low energy theorem

Weinberg 1966

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2} + \mathcal{O}(p^4) \qquad \Rightarrow \qquad T^{I=0} = \frac{2s - M_\pi^2}{F_\pi^2}$$

S wave projection

(I=0)

$$t_0^0(s) = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \qquad a_0^0 = t_0^0(4M_\pi^2) = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$

Low-energy theorem for $\pi\pi$ scattering

Some notation

$$\langle\pi^i\pi^j\ \mathrm{out}|\pi^k\pi^l\ \mathrm{in}\rangle = \delta^{ij}\delta^{kl}A(s,t,u)+\delta^{ik}\delta^{jl}A(t,u,s)+\delta^{il}\delta^{jk}A(u,t,s)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0}(s,t,u) = 3A(s,t,u)+A(t,s,u)+A(u,t,s)$$

Low energy theorem

Weinberg 1966

$$A(s,t,u)=\frac{s-M_\pi^2}{F_\pi^2}+\mathcal{O}(p^4) \qquad \Rightarrow \qquad T^{I=0}=\frac{2s-M_\pi^2}{F_\pi^2}$$

S wave projection

(I=2)

$$t_0^2(s)=\frac{2M_\pi^2-s}{32\pi F_\pi^2} \qquad a_0^2=t_0^2(4M_\pi^2)=\frac{-M_\pi^2}{16\pi F_\pi^2}=-0.045$$

Higher orders

Higher order corrections are suppressed by $\mathcal{O}(p^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

Gasser and Leutwyler (84)

Higher orders

Higher order corrections are suppressed by $\mathcal{O}(p^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

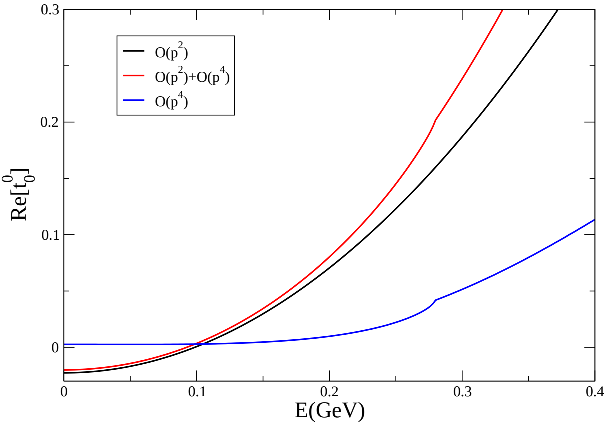
The reason for the rather large correction in a_0^0 is a chiral log

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{9}{2}\ell_\chi + \dots\right] \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{3}{2}\ell_\chi + \dots\right]$$

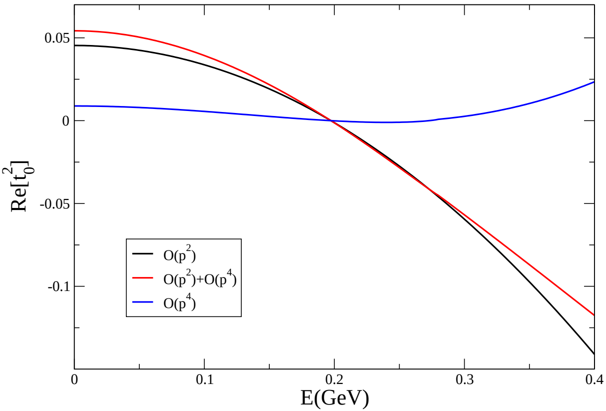
$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2} \ln \frac{\mu^2}{M_\pi^2}$$

Gasser and Leutwyler (84)

Higher orders



Higher orders



Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

Input: imaginary parts above 0.8 GeV
two subtraction constants, *e.g.* a_0^0 and a_0^2

Roy equations

Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

Input: imaginary parts above 0.8 GeV
two subtraction constants, *e.g.* a_0^0 and a_0^2
Output: the full $\pi\pi$ scattering amplitude below 0.8 GeV
Note: if a_0^0, a_0^2 are chosen within the universal band
the solution exist and is unique

Roy equations

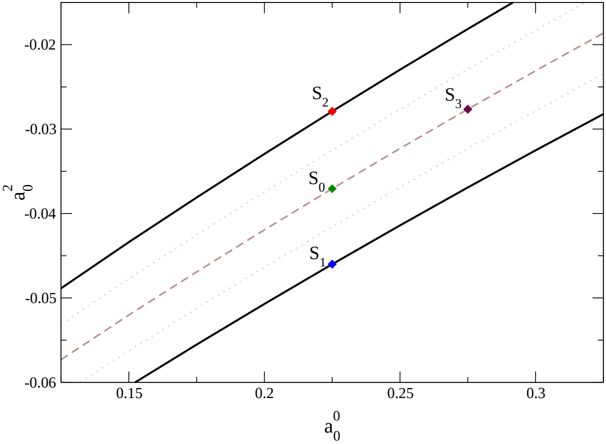
Unitarity effects can be calculated exactly using dispersive methods

Unitarity, analyticity and crossing symmetry $\equiv$ Roy equations

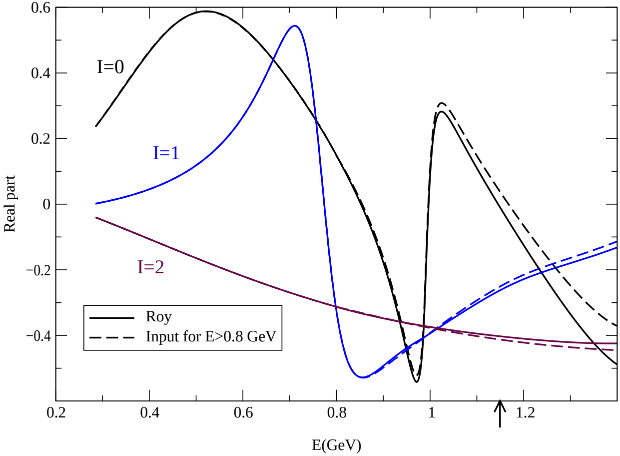
Input: imaginary parts above 0.8 GeV
two subtraction constants, *e.g.* a_0^0 and a_0^2
Output: the full $\pi\pi$ scattering amplitude below 0.8 GeV
Note: if a_0^0, a_0^2 are chosen within the universal band
the solution exist and is unique

Numerical solutions of the Roy equations
Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)

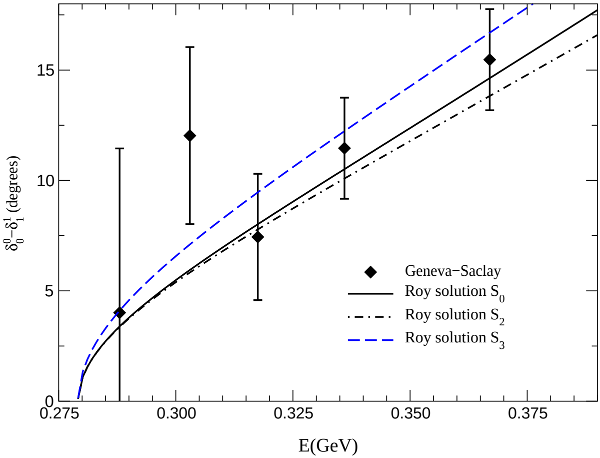
Numerical solutions



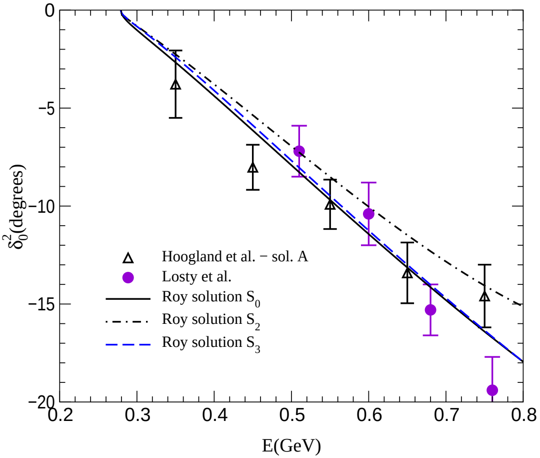
Numerical solutions



Numerical solutions



Numerical solutions



Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Subtracting the amplitude at threshold (a_0^0, a_0^2) is not mandatory

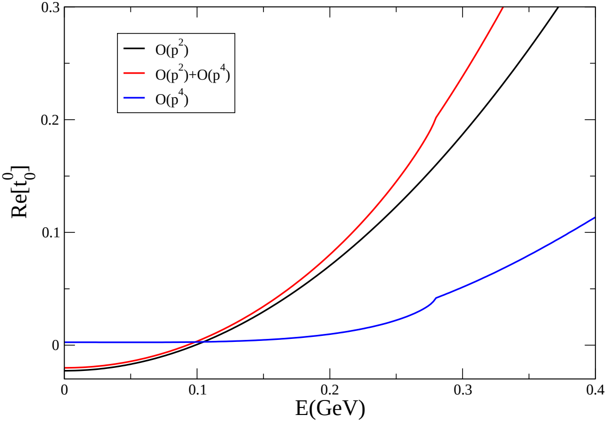
Combining CHPT and dispersive methods

In CHPT the two subtraction constants are predicted

Subtracting the amplitude at threshold (a_0^0, a_0^2) is not mandatory

The freedom in the choice of the subtraction point
can be exploited to use the chiral expansion
where it converges best, *i.e.* below threshold

Combining CHPT and dispersive methods



Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

$$
\begin{array}{rcccccc}
a_0^0 & = & 0.159 & \rightarrow & 0.200 & \rightarrow & 0.216 \\
10 \cdot a_0^2 & = & -0.454 & \rightarrow & -0.445 & \rightarrow & -0.445 \\
 & & p^2 & & p^4 & & p^6
\end{array}
$$

GC, Gasser and Leutwyler (01)

Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

$$\begin{array}{rcccc} a_0^0 & = & 0.159 \rightarrow & 0.200 \rightarrow & 0.216 \\ 10 \cdot a_0^2 & = & -0.454 \rightarrow & -0.445 \rightarrow & -0.445 \\ & & p^2 & p^4 & p^6 \end{array}$$

CHPT below threshold + Roy

$$\begin{array}{rcccc} a_0^0 & = & 0.197 \rightarrow & 0.2195 \rightarrow & 0.220 \\ 10 \cdot a_0^2 & = & -0.402 \rightarrow & -0.446 \rightarrow & -0.444 \end{array}$$

GC, Gasser and Leutwyler (01)

Final results

$$
\begin{aligned}
a_0^0 &= \quad 0.220 \pm 0.001 + 0.027\Delta_{r^2} - 0.0017\Delta\ell_3 \\
10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.04\Delta_{r^2} - 0.004\Delta\ell_3
\end{aligned}
$$

where

$$
\langle r^2 \rangle_s = 0.61\mathrm{fm}^2(1 + \Delta_{r^2}) \qquad \bar{\ell}_3 = 2.9 + \Delta\ell_3
$$

Final results

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.001 + 0.027\Delta_{r^2} - 0.0017\Delta\ell_3 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.04\Delta_{r^2} - 0.004\Delta\ell_3 \end{aligned}$$

where

$$\langle r^2 \rangle_s = 0.61\mathrm{fm}^2(1 + \Delta_{r^2}) \qquad \bar{\ell}_3 = 2.9 + \Delta\ell_3$$

Adding errors in quadrature $[\Delta_{r^2} = 6.5\%,\ \Delta\ell_3 = 2.4]$

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.005 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.01 \\ a_0^0 - a_0^2 &= 0.265 \pm 0.004 \end{aligned}$$

Final results

$$\begin{aligned} a_0^0 &= \quad 0.220 \pm 0.001 + 0.027\Delta_{r^2} - 0.0017\Delta\ell_3 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.04\Delta_{r^2} - 0.004\Delta\ell_3 \end{aligned}$$

where

$$\langle r^2 \rangle_s = 0.61\mathrm{fm}^2(1 + \Delta_{r^2}) \qquad \bar{\ell}_3 = 2.9 + \Delta\ell_3$$

Adding errors in quadrature $[\Delta_{r^2} = 6.5\%,\ \Delta\ell_3 = 2.4]$

$$\begin{aligned} a_0^0 &= \quad 0.220 \pm 0.005 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.01 \\ a_0^0 - a_0^2 &= \quad 0.265 \pm 0.004 \end{aligned}$$

Pelaez and Yndurain have criticized these results

Claim 1: our input above 1.4 GeV is not correct (PY 03)

The criticism has been answered (Caprini *et al.* 03)

Final results

$$\begin{aligned} a_0^0 &= \quad 0.220 \pm 0.001 + 0.027\Delta_{r^2} - 0.0017\Delta\ell_3 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.04\Delta_{r^2} - 0.004\Delta\ell_3 \end{aligned}$$

where

$$\langle r^2 \rangle_s = 0.61\mathrm{fm}^2(1 + \Delta_{r^2}) \qquad \bar{\ell}_3 = 2.9 + \Delta\ell_3$$

Adding errors in quadrature $[\Delta_{r^2} = 6.5\%,\ \Delta\ell_3 = 2.4]$

$$\begin{aligned} a_0^0 &= \quad 0.220 \pm 0.005 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.01 \\ a_0^0 - a_0^2 &= \quad 0.265 \pm 0.004 \end{aligned}$$

Pelaez and Yndurain have criticized these results

Claim 2: our calculation for $\langle r^2 \rangle_s$ is not correct (Y, 04)

The criticism has been answered (Ananthanarayan *et al.* 04)

Sensitivity to the quark condensate

The constant $\bar{\ell}_3$ appears in the chiral expansion of the pion mass

$$M_\pi^2 = 2B\hat{m}\left[1 + \frac{2B\hat{m}}{16\pi F_\pi^2}\bar{\ell}_3 + \mathcal{O}(\hat{m}^2)\right]$$

$$\hat{m} = \frac{m_u + m_d}{2} \qquad B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle$$

Sensitivity to the quark condensate

The constant $\bar{\ell}_3$ appears in the chiral expansion of the pion mass

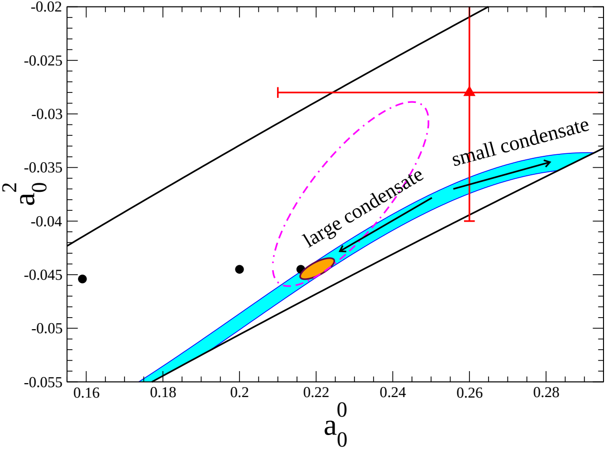
$$M_\pi^2 = 2B\hat{m}\left[1 + \frac{2B\hat{m}}{16\pi F_\pi^2}\bar{\ell}_3 + \mathcal{O}(\hat{m}^2)\right]$$

$$\hat{m} = \frac{m_u + m_d}{2} \qquad B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle$$

Its size tells us what fraction of the pion mass is given by the Gell-Mann–Oakes–Renner term

$$M_{\mathrm{GMOR}}^2 \equiv 2B\hat{m}$$

Sensitivity to the quark condensate



The E865 data on $K_{\ell 4}$ imply that

GC, Gasser and Leutwyler PRL (01)

$$M_{\text{GMOR}} > 94\% M_\pi$$

Conclusions

- The high precision in the prediction for the scattering lengths is obtained through a combined use of dispersive methods and chiral symmetry

Conclusions

- The high precision in the prediction for the scattering lengths is obtained through a combined use of dispersive methods and chiral symmetry
- The prediction relies on the assumption that the Gell-Mann–Oakes–Renner term dominates the pion mass

Conclusions

- The high precision in the prediction for the scattering lengths is obtained through a combined use of dispersive methods and chiral symmetry
- The prediction relies on the assumption that the Gell-Mann–Oakes–Renner term dominates the pion mass
- This assumption has been confirmed by the E865 data on K_{e4} decays

Conclusions

- The high precision in the prediction for the scattering lengths is obtained through a combined use of dispersive methods and chiral symmetry
- The prediction relies on the assumption that the Gell-Mann–Oakes–Renner term dominates the pion mass
- This assumption has been confirmed by the E865 data on K_{e4} decays
- Increasing the precision of the scattering length measurement will improve our knowledge of the QCD vacuum and will allow sensitive tests with lattice calculations