Comments on isospin breaking in $\pi\pi$ scattering

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1 Introduction

In the afternoon session of the recent NA48 meeting, there were several issues raised concerning isospin breaking in the $\pi\pi$ threshold amplitudes.

Here, we consider the threshold amplitudes a_x, a_{+0} etc. that occur in the scattering length expansion performed by Cabibbo and Isidori in Refs. [1, 2]. These quantities refer to a world where $M_{\pi^+} = 139.57$ MeV, M_{π^0}=134.98 MeV. In the work of Cabibbo and Isidori, these are the only isospin breaking effects taken into account. In particular, real and virtual photon emission is not considered. In order to have an estimate of the effect of isospin breaking in the threshold amplitudes, we consider in the following an effective lagrangian that describes exactly this situation. As this is only an approximation to the real situation, a tree level calculation seems appropriate.

2 Threshold amplitudes at $\alpha_{\rm QED} \neq 0$

In order to incorporate isospin breaking in the mass difference of the pions, while disregarding virtual and real photons in the amplitudes, we consider the effective lagrangian

$$\mathcal{L}_{\rm eff} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle + e^2 C \langle QUQU^\dagger\rangle \ , \tag{1}$$

with

$$\chi = 2B \ \mathrm{diag(m_u, m_d)} \ , \ \mathrm{Q} = \mathrm{diag}(2,-1)/3 \tag{2}$$

and

$$\begin{aligned} U &= \sigma + i\varphi/F \ ; \ \sigma = \sqrt{1_{2\times 2} - \varphi^2/F^2} \ , \\ \varphi &= \begin{pmatrix} \pi_0 & \sqrt{2}\pi_+ \\ \sqrt{2}\pi_- & -\pi_0 \end{pmatrix} \ . \end{aligned} \tag{3}$$

Expanding up to fourth order in the pion fields, one obtains

$$\begin{aligned} \mathcal{L}_{\rm eff} &= \frac{1}{2}(\partial_\mu\pi_0\partial^\mu\pi_0 - M^2\pi_0^2) + \partial_\mu\pi_+\partial^\mu\pi_- - M_\pi^2\pi_+\pi_- \\ &+ \frac{1}{8F^2}\left[\partial_\mu(\pi_0^2 + 2\pi_+\pi_-)\partial^\mu(\pi_0^2+2\pi_+\pi_-) - M^2(\pi_0^2+2\pi_+\pi_-)^2\right] \\ &+ \mathcal{O}(\varphi^6), \end{aligned} \tag{4}$$

where

$$M^2 = B(m_u + m_d) \ , \ M_\pi^2 = M^2 + \frac{2e^2 C}{F^2} \ . \tag{5}$$

In this approximation, $M(M_\pi)$ therefore denotes the neutral (the charged) pion mass,

$$M^2 = M_{\pi^0}^2 \ , \ M_\pi^2 = M_{\pi^+}^2 \ . \tag{6}$$

Next, we evaluate the matrix elements for $\pi\pi$ scattering,

$$\begin{aligned} T_{00;00} &= \pi^0\pi^0 \to \pi^0\pi^0 \ , \\ T_{+-;00} &= \pi^+\pi^- \to \pi^0\pi^0 \ , \\ T_{+0;+0} &= \pi^+\pi^0 \to \pi^+\pi^0 \ , \\ T_{+-;+-} &= \pi^+\pi^- \to \pi^+\pi^- \ , \\ T_{++;++} &= \pi^+\pi^+ \to \pi^+\pi^+ . \end{aligned} \tag{7}$$

Using the Condon-Shortley phase convention, we obtain at tree level

$$\begin{aligned} T_{00;00} &= \frac{M^2}{F^2} \ , \\ T_{+0;+0} &= \frac{t - M^2}{F^2} \ , \\ T_{+-;00} &= \frac{M^2 - s}{F^2} \ , \\ T_{+-;+-} &= \frac{s + t - 2M^2}{F^2} \ , \\ T_{++;++} &= \frac{t + u - 2M^2}{F^2} \ . \end{aligned} \tag{8}$$

Here, the normalization is such that the amplitude for the process

$$\pi^i(p_1)\pi^k(p_2) \to \pi^l(p_3)\pi^m(p_4)$$

is given by

$$\begin{aligned} T^{ik;lm} &= \delta^{ik}\delta^{ml} A(s,t,u) + \text{cycl.} \ , \\ A(s,t,u) &= \frac{s - M^2}{F^2} + \mathcal{O}(p^4) \ . \end{aligned} \tag{9}$$

Note that, in these kinematic variables, only M^2 occurs - the charged pion mass is hidden. Finally, we evaluate these amplitudes at the thresholds indicated in Eqs. (25)-(29) in [2], and divide the result by 32π. Furthermore, we use the phase convention of Cabibbo and Isidori, who choose the charged pion state to have a sign which differs from the Condon-Shortley one. Therefore, there is a sign change in $T_{+-;00}$. First, we present the expressions in the isospin symmetry limit $e = 0$. By convention, the pion mass is chosen to be the charged one. The isospin zero S-wave scattering lengths are given at this order in the chiral expansion by[1]

$$a_0 = \frac{7M_\pi^2}{32\pi F^2} \; , \; a_2 = -\frac{M_\pi^2}{16\pi F^2} \; , \tag{10}$$

and

$$\begin{aligned} \bar{a}_{00} &= (a_0 + 2a_2)/3 \; , \\ \bar{a}_{+0} &= a_2/2 \; , \\ \bar{a}_x &= (a_0 - a_2)/3 \; , \\ \bar{a}_{+-} &= (2a_0 + a_2)/6 \; , \\ \bar{a}_{++} &= a_2 \, . \end{aligned} \tag{11}$$

Here, the bar indicates that we consider the isospin symmetry limit and that we have used the charged pion mass as a reference point. Eq. (11) agrees with the relations Eqs. (25)-(29) in Ref. [2]. Finally, we display the threshold amplitudes at $e \neq 0$,

$$\begin{aligned} a_{00} &= \bar{a}_{00}(1-\epsilon) \; , \\ a_{+0} &= \bar{a}_{+0}(1-\epsilon) \; , \\ a_x &= \bar{a}_x(1+\frac{\epsilon}{3}) \; , \\ a_{+-} &= \bar{a}_{+-}(1+\epsilon) \; , \\ a_{++} &= \bar{a}_{++}(1-\epsilon) \; , \end{aligned} \tag{12}$$

where

$$\begin{aligned} \epsilon &= (M_\pi^2 - M^2)/M_\pi^2 \\ &= 6.5 \times 10^{-2} \, . \end{aligned} \tag{13}$$

[1] For historical reasons and for convenience, our scattering lengths are kept dimensionless.

Here, we have used Eq. (6). It is seen that the threshold amplitudes which occur in the scattering length expansion in Ref. [1, 2] are affected by substantial isospin breaking corrections.

3 Conclusion

In the decay rates for $K \to 3\pi$, one uses the physical amplitudes a_x, a_{+-} etc. Once these quantities have been determined from data, they can be related to the S-wave isospin zero scattering lengths using the corrections displayed in Eq. (12). If radiative corrections will be applied at a later stage, these relations must be adapted accordingly.

The correlation between a_0 and a_2 mentioned at the NA48 meeting by Colangelo (answering a question raised by Luigi DiLella) reads [see Ref. [3], Eq. (5)]

$$\begin{aligned} a_2 &= -0.0444 \pm 0.0008 + 0.236(a_0 - 0.22) \\ &\quad -0.61(a_0 - 0.22)^2 - 9.9(a_0 - 0.22)^3 \,. \end{aligned} \tag{14}$$

This relation holds for the case where the pion mass is identified with the charged one, $M_\pi = 139.57$ MeV. In case that this correlation is not built into the fitting procedure, it would be very useful to have the correlation matrix between the quantities a_x, a_{+-} determined from the fit.

References

[1] N. Cabibbo, Phys. Rev. Lett. **93** (2004) 121801 [arXiv:hep-ph/0405001].

[2] N. Cabibbo and G. Isidori, JHEP **0503** (2005) 021 [arXiv:hep-ph/0502130].

[3] G. Colangelo, J. Gasser and H. Leutwyler, Phys. Rev. Lett. **86** (2001) 5008 [arXiv:hep-ph/0103063].