Imporved Nagy-Soper method for numerical evaluation of one loop integrals

work in progress with

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Main Features

- Subtract each Feynman diagram individually
- Local MS-bar subtraction terms for UV, soft and collinear subtractions
- Feynman parametrize but do not carry out the loop integral
- Contour deformation
- Tested on simple examples

Subtract Feynman diagrams

The loop integral of a one loop subgraph, with n outgoing momenta

$$\Gamma(k_1, \ldots, k_n) \quad = \quad \int \frac{d^d l}{(2\pi)^d} \, \widetilde{\Gamma}(k_1, \ldots, k_n; l) \quad .$$

With $\overline{\mathrm{MS}}$ renormalization

$$[\Gamma]_{\mathrm{R}} = \lim_{\epsilon \longrightarrow 0} \left\{ \Gamma - [\Gamma]_{\mathrm{pole}} \right\} \quad ,$$

where

$$[\Gamma]_{\mathrm{pole}} = \frac{1}{\epsilon} \, \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \; \times \lim_{\epsilon \longrightarrow 0} \left[\epsilon \, \Gamma(\epsilon) \right] \quad .$$

We need local counter term for the UV subtraction free from IR singularities

$$[\Gamma]_{\mathrm{pole}} = \int \frac{d^d l}{(2\pi)^d} \, \widetilde{\Gamma}_{UV}(k_1, \ldots, k_n; l)$$

Local UV counter term

Recall the expression

$$\widetilde{\Gamma}(k_1,\ldots,k_n;l) \quad = \quad \frac{N(k_1,\ldots,k_n;l)}{(l_1^2+\mathrm{i}0)\cdots(l_n^2+\mathrm{i}0)} \quad .$$

define the UV divergent part N_{UV}, as

$$N(k_1,\ldots,k_n;l) = N_{UV}(k_1,\ldots,k_n;l) + \mathcal{O}(l^{2n-5}) \quad . \tag{1}$$

define

$$\widetilde{\Gamma}_{UV}(k_1,\ldots,k_n;l) \quad = \quad \frac{N_{UV}(k_1,\ldots,k_n;l)}{(l^2-\mu^2 e^{\lambda}+\mathrm{i}0)^n} \quad ,$$

μ : is the $\overline{\mathrm{MS}}$ renormalization scale

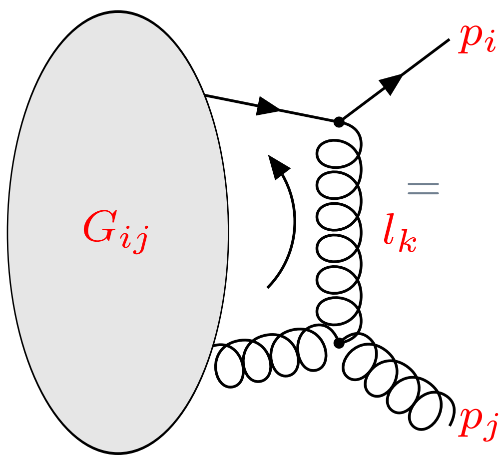
λ: apporpriate constant to get the MS counter term

$\widetilde{\Gamma}_{UV}$: free of infrared singularities and tt matches $\widetilde{\Gamma}$ for large l.

Local soft counter term

The integrand of a one-loop graph becomes singular when the momentum of an internal gluon loop line that connects to two external lines becomes soft.

$$\lim_{l_k\to 0}\frac{1}{E_{ij}(l_k)}$$



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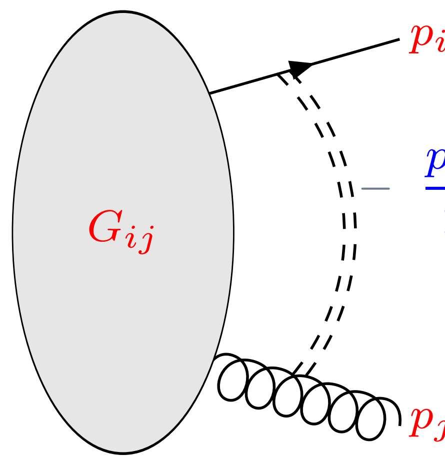
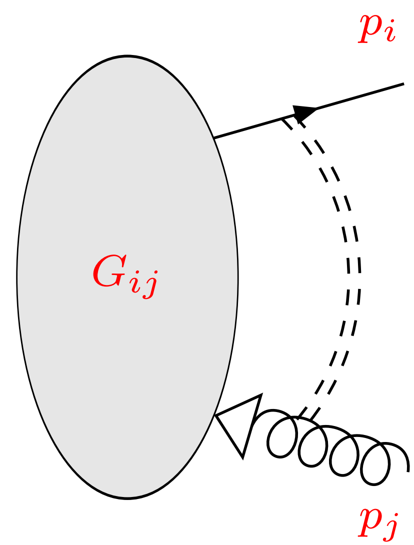


$$-\frac{p_i\cdot\varepsilon(p_j)}{2p_i\cdot p_j}$$



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The figure corresponds to the formula

$$\left|\widetilde{S}_{ij}(G;l_k;p_1,\ldots,p_m)\right\rangle = E_{ij}(l_k,p_i,p_j)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_j\,\boldsymbol{H}_{ij}\left|\mathcal{G}(G_{ij}(G);p_1,\ldots,p_m)\right\rangle \quad ,$$

$E_{ij}(l_k,p_i,p_j)$ is the eikonal factor

$$E_{ij}(l_k,p_i,p_j) = -\mathrm{i}g_s^2\mu^{2\epsilon}\frac{4p_i\cdot p_j}{(l_k^2+\mathrm{i}0)((l_k+p_i)^2+\mathrm{i}0)((l_k-p_j)^2+\mathrm{i}0)} \quad .$$

Note that this form is UV convergent. Performing the loop integral one gets

$$\mathcal{V}_{ij}^{\mathrm{soft}}(\epsilon) \equiv \int\frac{d^dl}{(2\pi)^d}\,E_{ij}(l) = \frac{\alpha_s}{4\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-2p_i\cdot p_j}\right)^\epsilon\left(\frac{2}{\epsilon^2}+\mathcal{O}(\epsilon)\right) \quad . \tag{2}$$

we add the integrals of these terms back in the form color correlated tree amplitudes

$$\sum_{\{i,j\}}\mathcal{V}_{ij}^{\mathrm{soft}}(\epsilon)\,\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left|\mathcal{M}_{\mathrm{tree}}(p_1,\ldots,p_m)\right\rangle \quad . \tag{3}$$

Local collinear counter terms

One-loop graphs have logarithmic infrared divergences that arise from integration regions in which the momentum on an internal loop line that connects to an external line becomes collinear with the momentum of the external line.

$$\begin{aligned} l_j &\to x\, p_i \\ -l_{j+1} &\to (1-x)\, p_i \quad , \end{aligned}$$

with $0 < x < 1$. Singular denominator factor $1/[l_j^2 (p_i - l_j)^2]$ with finite coefficient

$$\left| f_i^{C,0}(G; x; p_1, \ldots, p_m) \right\rangle = \lim_{l_j \to x\, p_i} l_j^2\, (p_i - l_j)^2 \left| \widetilde{\mathcal{G}}(G; l_1, \ldots, l_n; p_1, \ldots, p_m) \right\rangle \ .$$

The momentum subtraction away from the collinear limit is defined with the help of a second light like vector

$$n_i^\mu = -p_i^\mu + \frac{2\, p_i \cdot w}{w^2} w^\mu \ , \qquad w^\mu = \sum_{k \in \mathrm{final\ state}} p_k^\mu \ ; \quad x = \frac{l_j . n_i}{p_i . n_i}$$

...

$$\big|f_i^C(G;x;p_1,\dots,p_m)\big\rangle = \big|f_i^{C,0}(G;x;p_1,\dots,p_m)\big\rangle$$
$$-\frac{1}{x}\lim_{y\to 0} y\big|f_i^{C,0}(G;y;p_1,\dots,p_m)\big\rangle - \frac{1}{1-x}\lim_{y\to 1}(1-y)\big|f_i^{C,0}(G;y;p_1,\dots,p_m)\big\rangle \;.$$

$$f_{UV}(l_j,l_j-p_i) = \frac{1}{2}\left(\frac{-\mu^2 e}{l_j^2-\mu^2 e+\mathrm{i}0} + \frac{-\mu^2 e}{(l_j-p_i)^2-\mu^2 e+\mathrm{i}0}\right) \quad .$$

In summary, we subtract from the integrand for each graph G collinear subtraction terms

$$\big|\widetilde{C}_i(G;\{l\},p_1,\dots,p_m)\big\rangle \quad = \quad \frac{f_{UV}(l_j,l_j-p_i)}{(l_j^2+\mathrm{i}0)((p_i-l_j)^2+\mathrm{i}0)}$$
$$\times\int_0^1 dx\,\delta\!\left(x-\frac{l_j\cdot n_i}{p_i\cdot n_i}\right)\big|f_i^C(G;x;p_1,\dots,p_m)\big\rangle \;.$$

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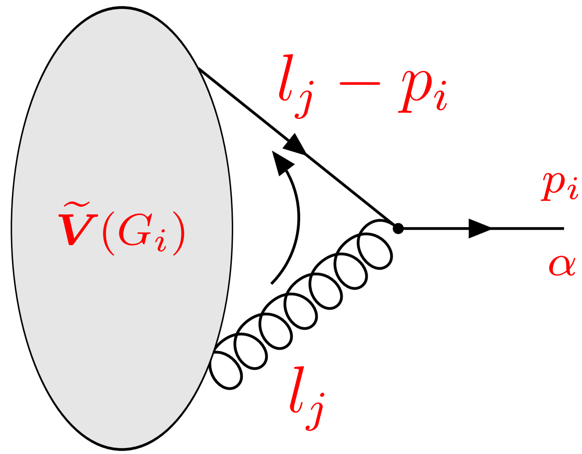
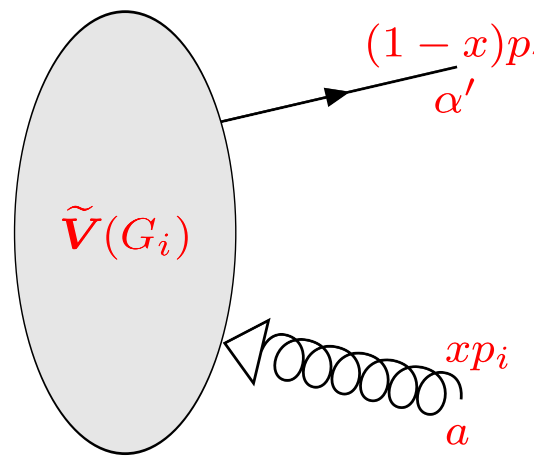
$$\lim_{l_j \to x p_i} l_j^2 (l_j - p_i)^2$$



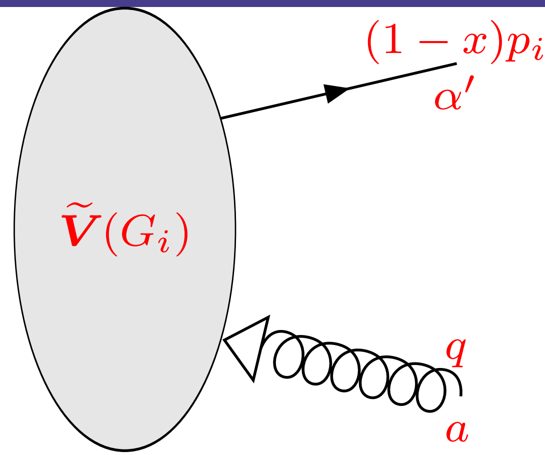
$=$

$$2\mathrm{i} g_s \mu^\epsilon \, T^a_{\alpha\alpha'} \, \frac{\sqrt{1-x}}{x}$$

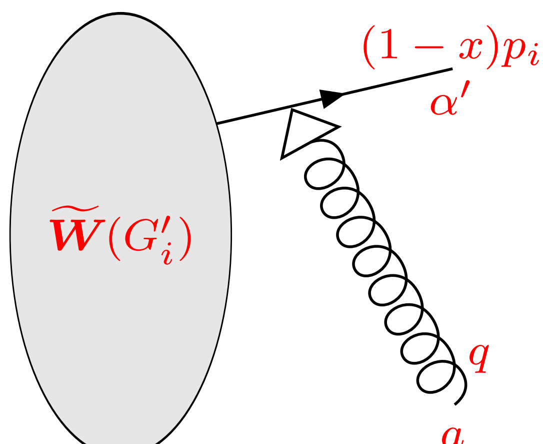


. . . summing over all diagrams

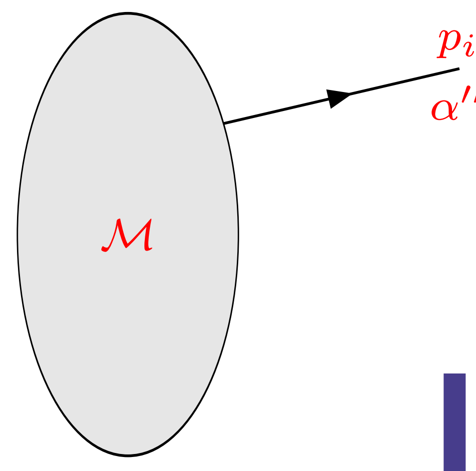
$$\frac{-1}{\sqrt{1-x}} \sum_{G_i \in C'_-} \lim_{q \to x p_i}$$



$$= \frac{-1}{\sqrt{1-x}} \lim_{q \to x p_i} \sum_{G'_i \in C_0}$$



$$= \mathrm{g}_s \mu^\epsilon \, T^a_{\alpha' \alpha''}$$



. . . integrating the loop momentum

$$\sum_{G\in C} \mathcal{C}_i(G;p_1,\ldots,p_m)^b$$

$$= \int \frac{d^d l_j}{(2\pi)^d} \, \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \int_0^1 dx \, \delta\left(x - \frac{l_j \cdot n_i}{p_i \cdot n_i} \right) \mathrm{i}g_s \, \mu^\epsilon T^a_{bc}$$

$$\times \left\{ \left[\frac{(2-x)}{x} - \frac{2}{x} - \frac{0}{1-x} \right] g_s \mu^\epsilon T^a_{cd} \, \mathcal{M}(p_1,\ldots,p_i,\ldots,p_m)^d \right\}$$

$$= -\mathrm{i}g_s^2 C_A \, \mu^{2\epsilon} \int \frac{d^d l_j}{(2\pi)^d} \, \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \mathcal{M}(p_1,\ldots,p_i,\ldots,p_m)^b$$

$$= \frac{\alpha_s}{4\pi} C_A \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon) \right) \mathcal{M}(p_1,\ldots,p_i,\ldots,p_m)^b \quad .$$

Summing over all graphs the collinear subtraction terms associated with an external gluon line with label i, we get a simple singular factor times the tree level amplitude $\mathcal{M}$.

Summary

- We generate local subtraction terms for the integrands of every graphs. The subtraction terms are designed to have the same effect as standard $\overline{\rm MS}$ renormalization. The UV, collinear and soft divergences of the integrals are removed.
- Sum over the subtracted terms of all graphs, carry out the loop integral and add back the answer which has the form as a result one gets

$$\mathbf{I}^V(\epsilon)\big|\mathcal{M}_{\rm tree}\,(p_1,\ldots,p_m)\big\rangle\;\;,$$

where

$$\mathbf{I}^V(\epsilon) = \frac{\alpha_s}{4\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left(\frac{\mu^2}{-2p_i\cdot p_j}\right)^\epsilon - \frac{1}{\epsilon}\sum_{i=1}^{m}\gamma_i\right)\;\;,$$

where the γ_i factors are

$$\gamma_q = \gamma_{\bar q} = \frac{3}{2}C_F\;\;,\qquad \gamma_g = \frac{11}{6}C_A - \frac{4}{6}T_R n_f\;\;.$$

...

$$\sigma = \sigma^{\mathrm{LO}\{m-n_I\}} + \sigma_{\mathrm{sub}}^{\mathrm{NLO}\{m+1-n_I\}} + \sigma_{\mathrm{sub}}^{\mathrm{NLO}\{m-n_I\}} + \Delta\sigma^{\mathrm{NLO}\{m-n_I\}} \ .$$

$$\begin{aligned}
\sigma_{\mathrm{sub}}^{\mathrm{NLO}\{m-n_I\}} \quad = \quad & \sum_G \int d\Phi^{m-n_I}\; F_J^{(m-n_I)}(p_1,\dots,p_m) \int \frac{d^4 l}{(2\pi)^4} \times 2\,\mathrm{Re}\Big\{ \\
& \Big\langle \mathcal{M}_{\mathrm{tree}}\,(p_1,\dots,p_m)\, \Big| \widetilde{\mathcal{G}}(G;\{l\};p_1,\dots,p_m) \Big\rangle \\
& - \Big\langle \mathcal{M}_{\mathrm{tree}}\,(p_1,\dots,p_m)\, \Big| \widetilde{\mathcal{R}}(G;\{l\};p_1,\dots,p_m) \Big\rangle \\
& - \sum_{\{i,j\}\in I_S(G)} \Big\langle \mathcal{M}_{\mathrm{tree}}\,(p_1,\dots,p_m)\, \Big| \widetilde{\mathcal{S}}_{ij}(G;\{l\};p_1,\dots,p_m) \Big\rangle \\
& - \sum_{i\in I_C(G)} \Big\langle \mathcal{M}_{\mathrm{tree}}\,(p_1,\dots,p_m)\, \Big| \widetilde{\mathcal{C}}_i(G;\{l\};p_1,\dots,p_m) \Big\rangle \Big\} \ .
\end{aligned}$$

...

$$\Delta\sigma^{\mathrm{NLO}\{m-n_I\}} = \int d\Phi^{m-n_I}\, F_J^{(m-n_I)}(p_1,\dots,p_m)$$
$$\times 2 \lim_{\epsilon\to 0} \left\langle \mathcal{M}_{\mathrm{tree}}\,(p_1,\dots,p_m)\,\middle|\mathbf{I}^V(\epsilon)+\mathbf{I}^R(\epsilon)\middle|\mathcal{M}_{\mathrm{tree}}\,(p_1,\dots,p_m)\,\right\rangle \ .$$

Does it work??

- Integration countour for loop integral ?
- Efficient sampling?
- Sum over virtual graphs numerically?

$4 + N$ dimensional approach

$$I = \Gamma(N) \int_0^1 dx^1 \cdots \int_0^1 dx^N \; \delta\Big(\sum x^i - 1\Big) \int d^4 l \, \frac{N(l + \sum_i x^i Q_i)}{[l^2 - \Lambda^2(\vec{x}) + i0]^N} \quad ,$$

where

$$\Lambda^2(\vec{x}) = -\frac{1}{2} \sum_{ij} x^i x^j S_{ij} \, ,$$

with (using $\sum x^i = 1$)

$$S_{ij} = (Q_i - Q_j)^2 - m_i^2 - m_j^2 \, .$$

- Do not perform the momentum integral analytically. It would produce many terms.
- Singular surface at the zeros of one quadratic funtion
- Subtraced functions have no singularities but have peaks around the original singular regions.
- Obtain smooth functions with contour deformations

Pinch singularities

- No UV divergence or anomalous thresholds. Subtracted function is convergent as $l \to \infty$
- Singularities come from the l integral at $l = 0$ when $\Lambda^2(\vec{x}) = 0$. These can be avoided taking $l \to l_c + i\kappa$
- If $\Lambda^2(\vec{x}) = 0$ along the boundary of the x-integration region(one or more of the x^i vanishes) we get end point singularities
- soft singularities: all but one x^ivanish
- collinear singularities: all but two adjacent x^i vanish
- expand $\Lambda^2(\vec{x})$ around the soft and collinear region
- Deform also the contour of the x-integration so that $\Lambda^2(x)$ does not vanish except at the boundaries

Toy example

- Order (α_s) QCD correction to the e^+e^- annihilation to hadrons. You get the answer numerically in two minutes with 1% accuracy.
- We are just working on three jet production. One numerical solutions was already worked out by Dave Soper
- Our first test is encouraging