

Construction of a relativistic field theory

Lagrangian

$$L = T - V$$

(Nonrelativistic mechanics)

Action

$$S = \int_{t_1}^{t_2} L dt$$

- Classical path ... minimises action

Feynman lectures

- Quantum mechanics ... sum over all paths with amplitude $\propto e^{iS/\hbar}$

Lagrangian invariant under all the symmetries of nature

-makes it easy to construct viable theories

Lagrangian formulation of the Klein Gordon equation

$L = \int L d^3x$, L lagrangian density

Klein Gordon field $\phi(x)$

$$L = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$



T



V

Manifestly Lorentz invariant

Dimensionality:

$$\frac{\partial L}{\partial \phi} - \partial^\mu \frac{\partial L}{\partial (\partial^\mu \phi)} = 0 \quad \text{Euler Lagrange equation}$$

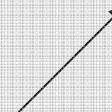
Euler Lagrange equs

$$S = \int_{t_1}^{t_2} L dt = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x$$

Principle of least action :

$$\begin{aligned} 0 = \delta S &= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \delta (\partial^\mu \phi) \right\} \\ &= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \right) \delta \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \delta \phi \right) \right\} \end{aligned}$$

0 (surface integral)





$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} = 0$$

Euler Lagrange equation

Lagrangian formulation of the Klein Gordon equation

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$$L = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$



T

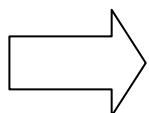


V

Manifestly Lorentz invariant

$$\frac{\partial L}{\partial \phi} - \partial^\mu \frac{\partial L}{\partial (\partial^\mu \phi)} = 0$$

Euler Lagrange equation



$$(\partial_\mu \partial^\mu + m^2) \phi = 0$$

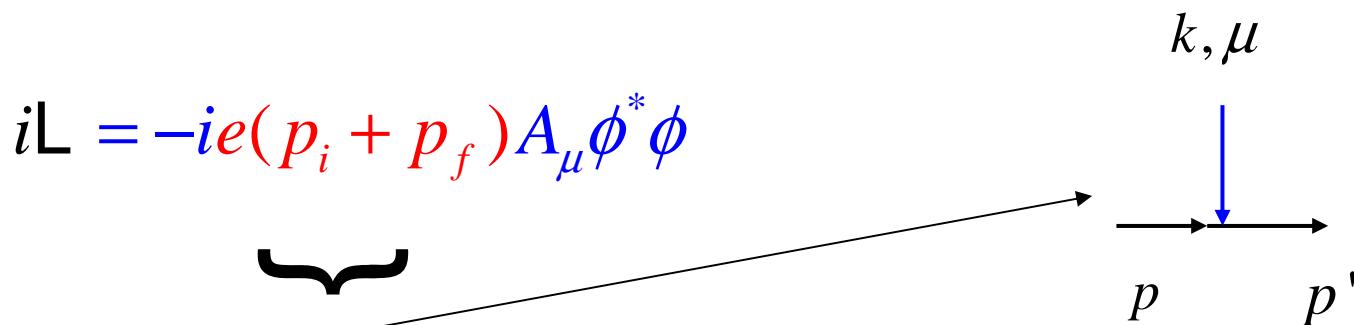
Klein Gordon equation

The Lagrangian and Feynman rules

Associate with the various terms in the Lagrangian a set of propagators and vertex factors

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.
- The remaining terms in the Lagrangian are associated with interaction vertices. The Feynman vertex factor is just given by the coefficient of the corresponding term in $i\mathcal{L}$

e.g. $\left((\partial_\mu - ieA_\mu)\phi(x) \right)^\dagger (\partial^\mu - ieeA^\mu)\phi(x) \rightarrow -ieA_\mu(\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi)$



New symmetries

$$\mathcal{L} = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \rightarrow e^{i\alpha} \phi(x)$...an Abelian (U(1)) gauge symmetry

A symmetry implies a conserved current and charge.

e.g. Translation \rightarrow Momentum conservation

Rotation \rightarrow Angular momentum conservation

What conservation law does the U(1) invariance imply?

Noether current

$$\mathcal{L} = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \rightarrow e^{i\alpha} \phi(x)$...an Abelian (U(1)) gauge symmetry

$$\begin{aligned} 0 = \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + (\phi \leftrightarrow \phi^\dagger) \\ &= i\alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \phi + i\alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \phi \right) - (\phi \leftrightarrow \phi^\dagger) \end{aligned}$$

i $\alpha\phi$ *i $\alpha\partial_\mu\phi$*
0 (Euler lagrange eqs.)



$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^\dagger)} \phi^\dagger \right)$$

Noether current

The Klein Gordon current

$$\mathcal{L} = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\psi(x) \rightarrow e^{i\alpha} \psi(x)$...an Abelian (U(1)) gauge symmetry

$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \phi^\dagger \right)$$

$$j_\mu^{KG} = -ie \left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^* \right)$$

This is of the form of the electromagnetic current we used for the KG field

The Klein Gordon current

$$\mathcal{L} = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\psi(x) \rightarrow e^{i\alpha} \psi(x)$...an Abelian (U(1)) gauge symmetry

$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \phi^\dagger \right)$$

$$j_\mu^{KG} = -ie(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$$

This is of the form of the electromagnetic current we used for the KG field

$$Q = \int d^3x j^0 \quad \text{is the associated conserved charge}$$

Suppose we have two fields with different U(1) charges :

$$\phi_{1,2}(x) \rightarrow e^{i\alpha Q_{1,2}} \phi_{1,2}(x)$$

$$\begin{aligned} \mathcal{L} = & \left(\partial_\mu \phi_1(x) \right)^\dagger \partial^\mu \phi_1(x) - m^2 \phi_1(x)^\dagger \phi_1(x) \\ & + \left(\partial_\mu \phi_2(x) \right)^\dagger \partial^\mu \phi_2(x) - m^2 \phi_2(x)^\dagger \phi_2(x) \end{aligned}$$

..no cross terms possible (corresponding to charge conservation)

U(1) local gauge invariance and QED

$$\phi(x) \rightarrow e^{i\alpha(x)Q} \phi(x)$$

$$L = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) \text{ not invariant due to derivatives}$$

$$\partial_\mu \phi \rightarrow \partial_\mu e^{i\alpha(x)Q} \phi = e^{i\alpha(x)Q} \partial_\mu \phi + iQ e^{i\alpha(x)Q} \phi \partial_\mu \alpha(x)$$

To obtain invariant Lagrangian look for a modified derivative transforming covariantly

$$D_\mu \phi \rightarrow e^{i\alpha(x)} D_\mu \phi$$

Need to introduce a new vector field $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$

$$D_\mu = \partial_\mu - iQA_\mu$$

$$\phi(x) \rightarrow e^{iQ\alpha(x)} \phi(x)$$

$$D_\mu \phi \rightarrow e^{i\alpha(x)} D_\mu \phi$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha$$

$$\mathcal{L} = (D_\mu \phi(x))^\dagger D^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) \quad \text{is invariant under local U(1)}$$

Note : $\partial_\mu \rightarrow D_\mu = \partial_\mu - iQA_\mu$ is equivalent to $p^\mu \rightarrow p^\mu + eA^\mu$

universal coupling of electromagnetism follows from local gauge invariance

$$\mathcal{L} = (\partial_\mu \phi(x))^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) - j^\mu A_\mu + O(e^2)$$

The electromagnetic Lagrangian

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu}, \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha$$

$$\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\mathcal{L}^{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$M^2 A^\mu A_\mu$ Forbidden by gauge invariance

The Euler-Lagrange equations give Maxwell equations !

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu A^\nu)} = 0$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

\equiv

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$$

The photon propagator

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial^\mu A_\mu) = j^\nu$$

Choose as

$$-\frac{1}{\xi} \partial^\mu A_\mu$$

(gauge fixing)

Gauge ambiguity

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha$$

$$\partial^\mu A_\mu \rightarrow \partial^\mu A_\mu + \partial^\mu \alpha$$

i.e. with suitable “gauge” choice of α (“ ξ ” gauge) want to solve

$$\partial_\mu \partial^\mu A^\nu - (1 - \frac{1}{\xi}) \partial^\nu (\partial_\mu A^\mu) \equiv (g^{\nu\mu} \partial^2 - (1 - \frac{1}{\xi}) \partial^\nu \partial_\mu) A^\mu = j^\nu$$

In momentum space

$$\left(g^{\mu\nu} p^2 - (1 - \frac{1}{\xi}) p^\mu p^\nu \right)^{-1} = \frac{i}{p^2} \left(-g_{\mu\nu} + (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right)$$

(‘t Hooft Feynman gauge $\xi=1$)

The inclusion of fermions

Weyl spinors

$$(\frac{1}{2}, 0) \quad (0, \frac{1}{2})$$

$$\psi_L \quad \psi_R$$

Dirac spinor

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

2-component spinors of SU(2)

Rotations and Boosts

$$\psi_{L(R)} \rightarrow S_{L(R)} \psi_{L(R)}$$

$$S_{L(R)} = e^{i\frac{\sigma}{2}\cdot\omega} : \text{Rotations}$$

$$S_{L(R)} = e^{\pm\frac{\sigma}{2}\cdot v} : \text{Boosts}$$

Dirac Gamma matrices

$$\gamma_0 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Weyl basis

$$\gamma_\mu \equiv (\gamma_0, \gamma_i) \text{ 4-vector}$$

$$\psi_{L(R)} = \frac{1}{2}(1 \mp \gamma_5)\psi$$

The Dirac equation

Fermions described by 4-cpt Dirac spinors ψ

- $\psi^\dagger \gamma^0 \psi \equiv \bar{\psi} \psi$ Lorentz invariant
- New 4-vector γ_μ

The Lagrangian

$$L = i\bar{\psi} \gamma_\mu \partial^\mu \psi - m\bar{\psi} \psi$$
 Dimension?

From Euler Lagrange equation obtain the Dirac equation

$$(i\gamma_\mu \partial^\mu - m)\psi = 0$$

U(1) symmetry

$$\psi \rightarrow e^{i\alpha} \psi \quad \rightarrow$$

$$j^\mu = -e\bar{\psi} \gamma_\mu \psi$$

Feynman rules

