New methods in QCD

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HERA-LHC workshop,

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Overview

- In a recent paper **Witten** made a striking proposal to relate **perturbative gauge theory amplitudes** to **topological string theory in twistor space**

Witten, hep-th/0312171

$\Rightarrow$ Advance in calculating **tree** amplitudes in massless gauge theories:

Cachazo, Svrcek and Witten, hep-th/0403047

Amplitudes constructed from **scalar propagators** and tree-level maximal helicity violating (MHV) amplitudes which are interpreted as new **scalar vertices**

$\Rightarrow$ New type of **on-shell** recursion relations

Britto, Cachazo and Feng, hep-th/0412308

$\Rightarrow$ Recent developments in computing **one-loop** amplitudes in $\mathcal{N}=4$ SuperYang Mills theory (as well as $\mathcal{N}=1$ and maybe even QCD)

State of play circa 2003

Multi-jet production at the LHC using HELAC/PHEGAS

Draggiotis, Kleiss, Papadopoloulos

# of jets	2	3	4	5	6	7	8
# of dist.processes	10	14	28	36	64	78	130
total # of processes	126	206	621	861	1862	2326	4342
$\sigma(nb)$	-	91.41	6.54	0.458	0.030	0.0022	0.00021
% Gluonic	-	45.7	39.2	35.7	35.1	33.8	26.6

- The number of Feynman diagrams for an n gluon process increases very quickly with n

$\Rightarrow$ for the 10 gluon amplitude there are 10,525,900 diagrams

$\Rightarrow$ Feynman diagrams very inefficient for many legs

- Control the quantum numbers of the scattering particles

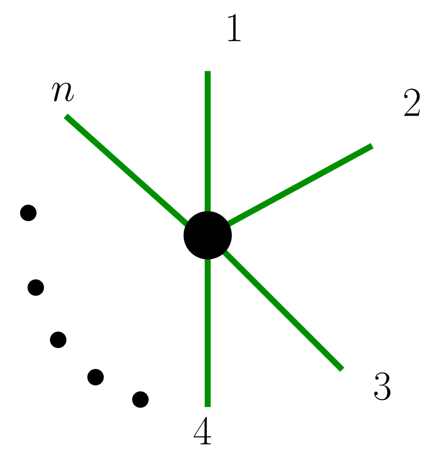
Feynman diagrams - Colour Ordered Amplitudes

$$\mathcal{A}_n(1,\ldots,n) = \sum_{perms} Tr(T^{a_1}\ldots T^{a_n}) A_n(1,\ldots,n)$$

Colour-stripped amplitudes A_n: cyclically ordered

Order of external gluons fixed

The subamplitudes A_n have nice properties in the infrared limits.

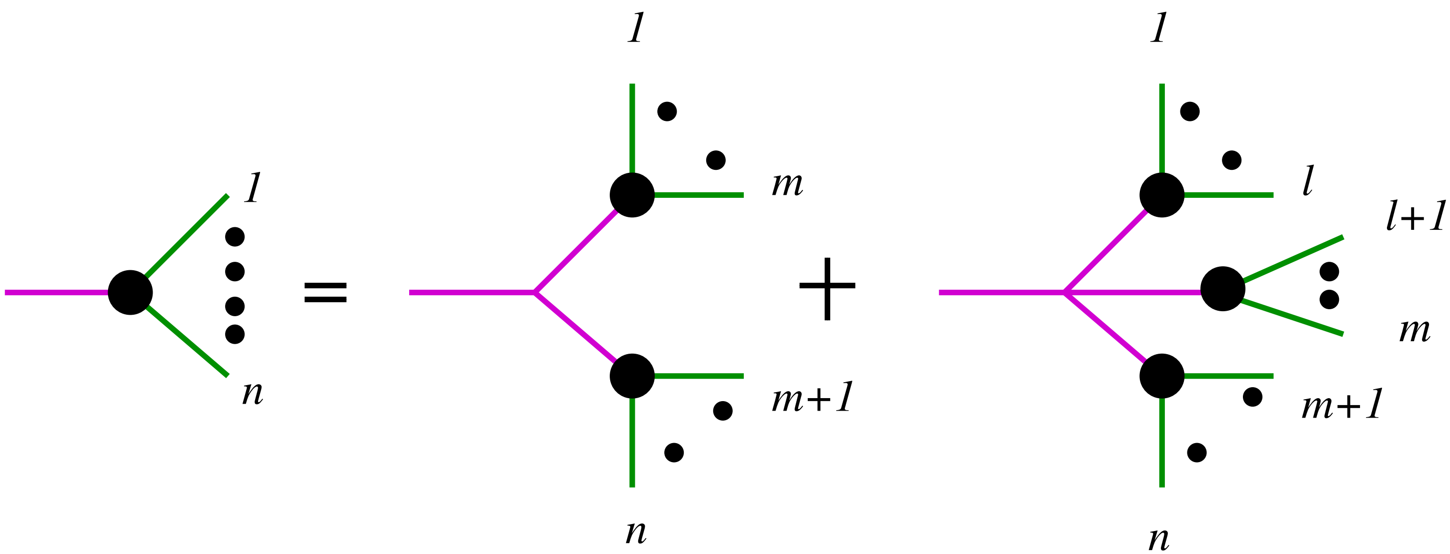


Can reconstruct the full amplitude $\mathcal{A}_n$ from A_n.
In the large N limit,

$$|\mathcal{A}_n(1,\ldots,n)|^2 \sim N^{n-2} \sum_{perms} |A_n(1,\ldots,n)|^2$$

Feynman diagrams : Recursion relations

Full amplitudes can be built up from simpler amplitudes with fewer particles



Purple gluons are off-shell, green gluons are on-shell.
This is a recursion relation built from off-shell currents.

Berends, Giele

Particularly suited to numerical solution

ALPGEN, HELAC/PHEGAS

Feynman diagrams : Spinor Helicity Formalism

In four dimensions, write massless vector

$$p_{a\dot{a}} \equiv p_\mu \sigma^\mu_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

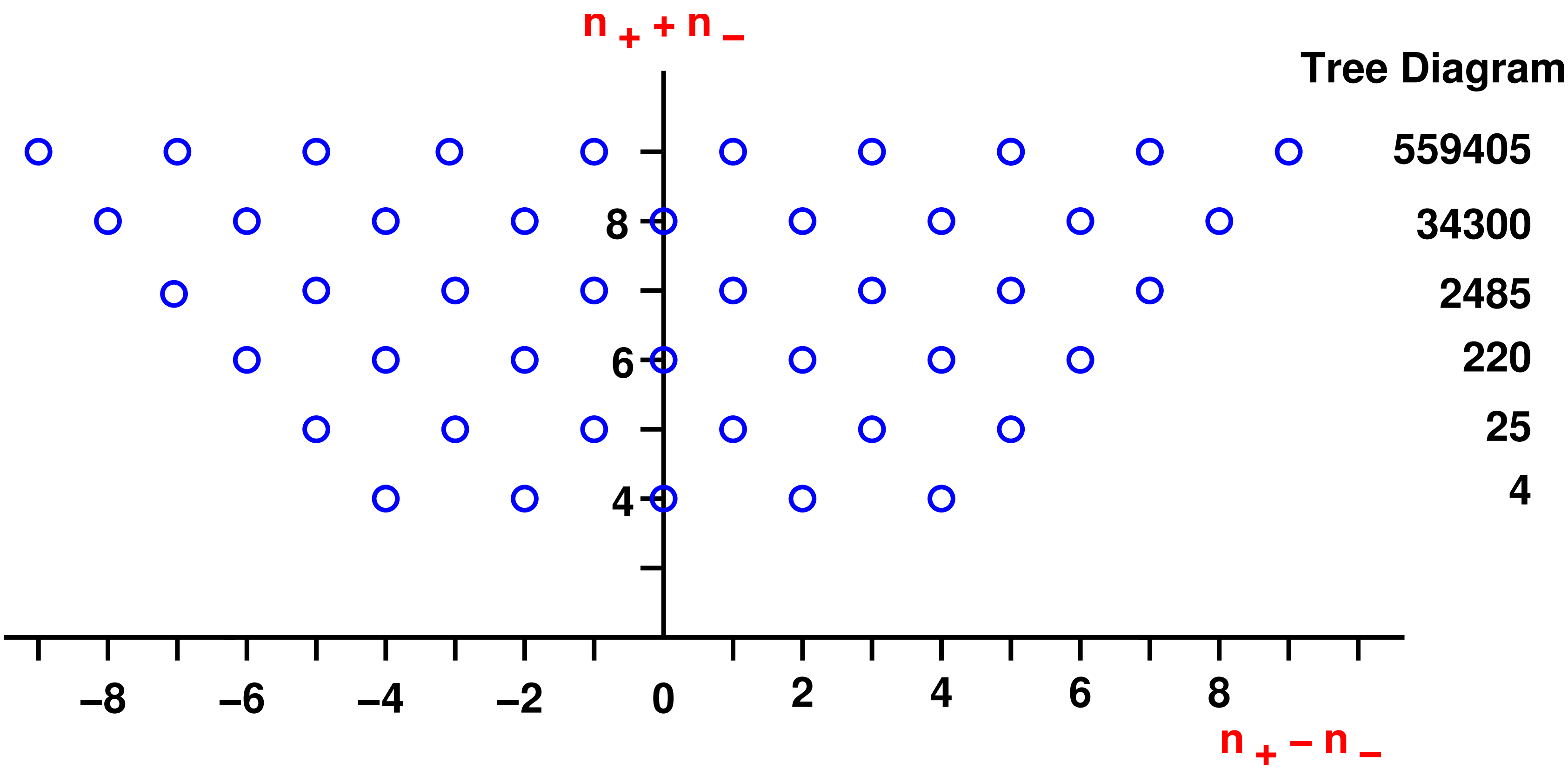
where λ_a and $\tilde{\lambda}_{\dot{a}}$ are commuting Weyl spinors of positive and negative chirality.
Spinor products are

$$\begin{aligned}
\langle \lambda_i, \lambda_j \rangle &= \epsilon^{ab} \lambda_{ia} \lambda_{jb} = \langle ij \rangle = -\langle ji \rangle = \bar{u}^-(i) u^+(j) \\
\langle \tilde{\lambda}_i, \tilde{\lambda}_j \rangle &= -\epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}} = [ij] = \langle ji \rangle^* = \bar{u}^+(i) u^-(j) \\
s_{ij} &= (p_i + p_j)^2 = 2\, p_{i\mu} p_j^\mu = \langle ij \rangle [ji]
\end{aligned}$$

Gauge vectors: η is reference momentum $\leftrightarrow$ gauge choice

$$\varepsilon^-_{ia\dot{a}} = \frac{\lambda_{ia} \tilde{\eta}_{\dot{a}}}{[\tilde{\lambda}_i \tilde{\eta}]} \qquad \varepsilon^+_{ia\dot{a}} = \frac{\eta_a \tilde{\lambda}_{i\dot{a}}}{\langle \eta \lambda_i \rangle}$$

Gluonic helicity amplitudes



Each row describes scattering with n_+ positive helicities and n_- negative helicities.
Each circle represents an allowed helicity configuration - from all positive on the right to all negative on the left

Gluonic helicity amplitudes

For example, the result of computing the 25 diagrams for the five-gluon process yields

$$A_5(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0$$

$$A_5(1^-, 2^-, 3^+, 4^+, 5^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

In fact, for n point amplitudes,

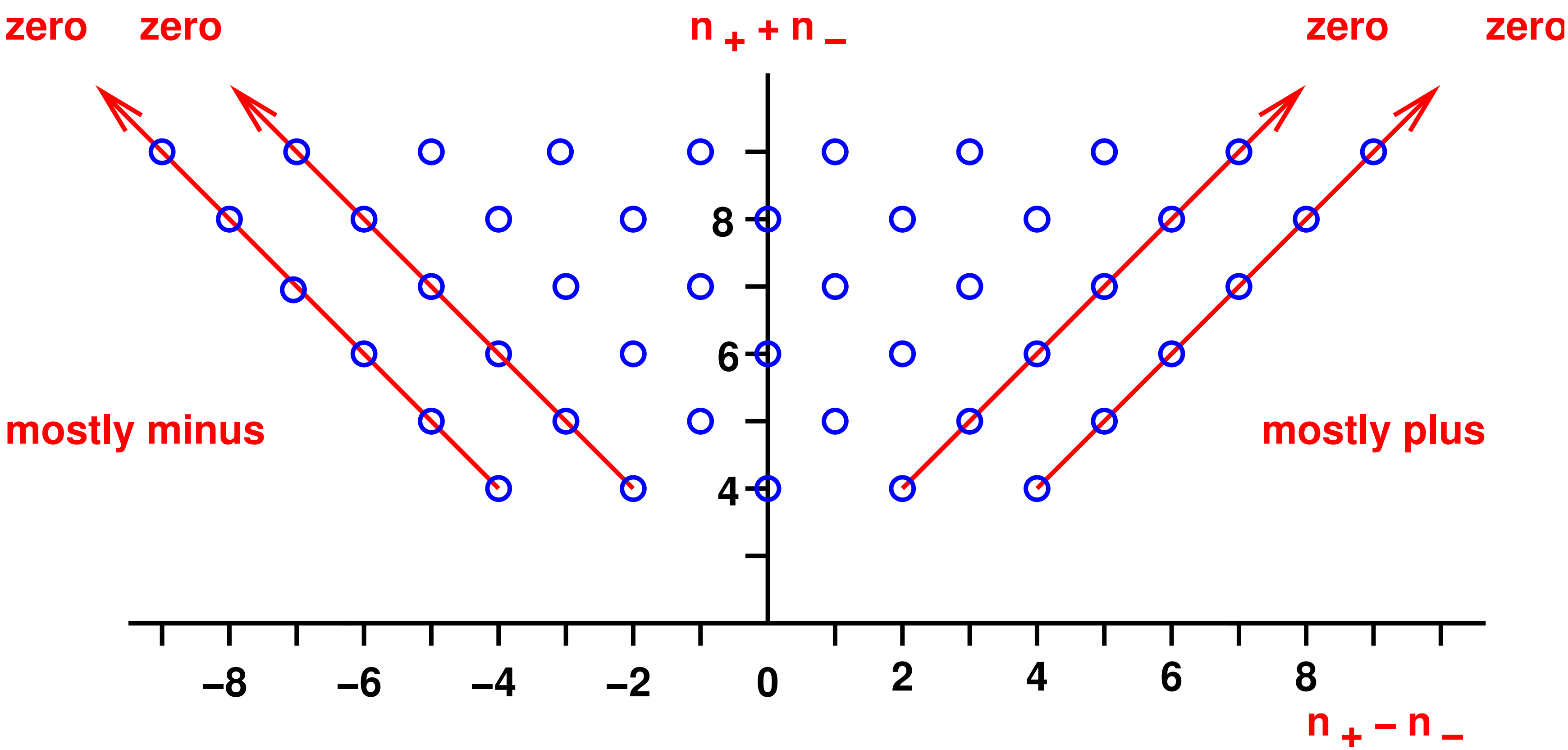
$$A_n(1^\pm, 2^+, 3^+, , \ldots, n^+) = 0$$

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

Maximally helicity violating (MHV) amplitudes

Parke, Taylor; Berends, Giele

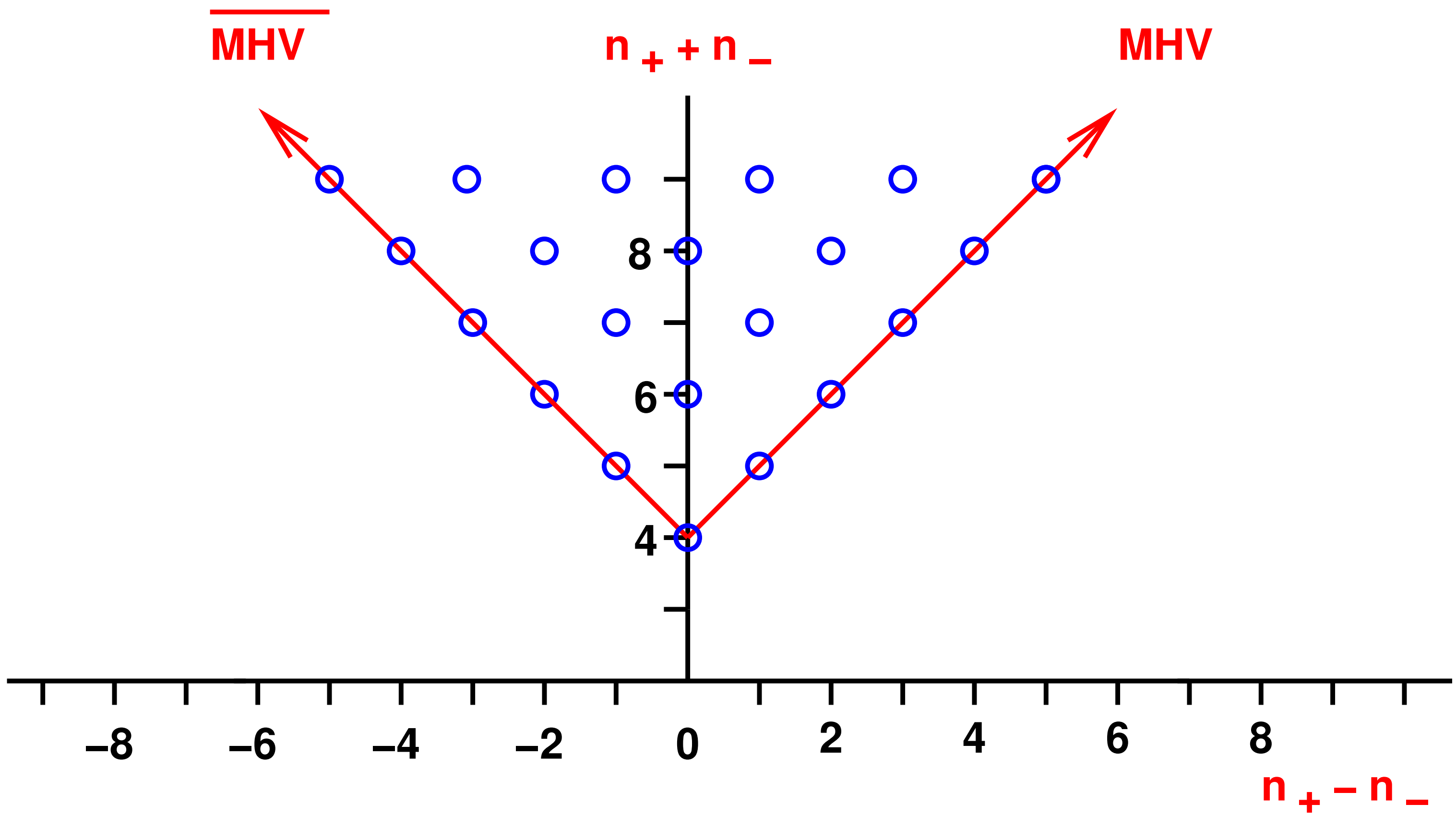
Gluonic helicity amplitudes



$$A_n(1^{\pm}, 2^{+}, 3^{+}, , \ldots, n^{+}) = 0$$

effective tree-level supersymmetry

Gluonic helicity amplitudes



$$A_n(1^-, 2^-, 3^+, \ldots, n^+) \quad = \quad \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Specific helicity amplitudes

For phenomenological purposes, all possible helicity amplitudes are needed - and which are usually much more complicated. For example, the 220 six gluon diagrams contributing to NMHV amplitudes (3- and 3+ helicities) can be written as

$$A_6 = 8g^4 \left[\frac{\alpha^2}{s_{123}s_{12}s_{23}s_{34}s_{45}s_{56}} + \frac{\beta^2}{s_{234}s_{23}s_{34}s_{45}s_{56}s_{61}} + \frac{\gamma^2}{s_{345}s_{34}s_{45}s_{56}s_{61}s_{12}} + \frac{s_{123}\beta\gamma + s_{234}\gamma\alpha + s_{345}\alpha\beta}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}} \right]$$

where for $A_6(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$,

$$\alpha = 0, \qquad \beta = [23]\langle 56\rangle\langle 1|\not{2}+\not{3}|4\rangle, \qquad \gamma = [12]\langle 45\rangle\langle 3|\not{1}+\not{2}|6\rangle,$$

Hidden structure is uncovered in **twistor space**

Twistor Space

Twistor space:

Penrose, 1967

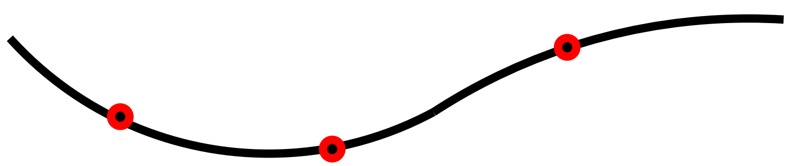
Amplitudes in twistor space obtained by Fourier transform with respect to positive helicity spinors,

$$\tilde{A}(\lambda_i, \mu_{\dot{i}}) = \int \prod_i \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} \exp\left(i \sum_j \mu_j^{\dot{a}} \tilde{\lambda}_{j\dot{a}} \right) A(\lambda_i, \tilde{\lambda}_i)$$

Witten observed that in twistor space external points lie on certain algebraic curves

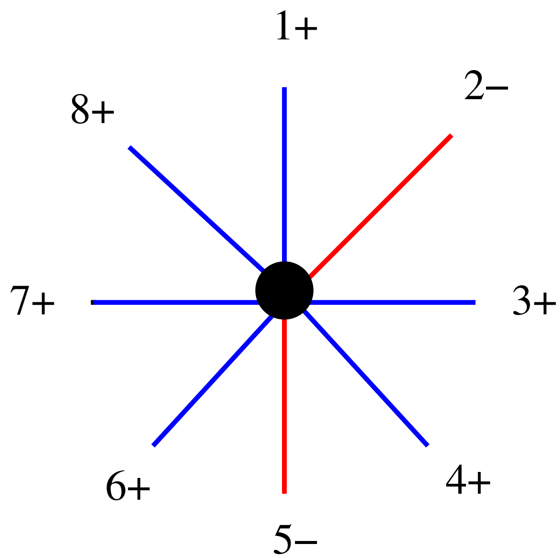
$\Rightarrow$ degree of curve is related to the number of negative helicities and loops

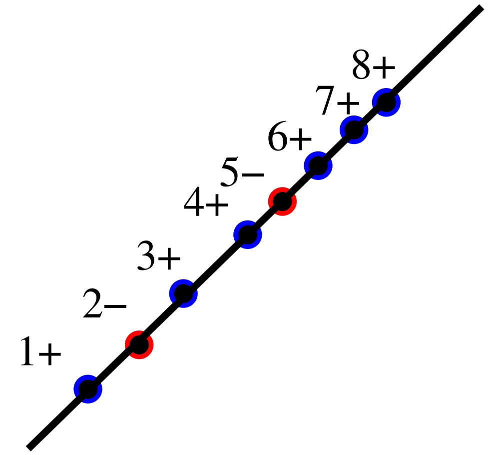
$$d = n_- - 1 + l$$



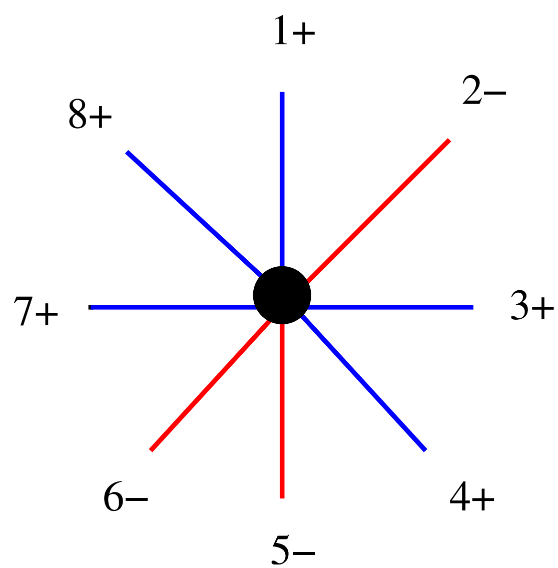
Twistor Space

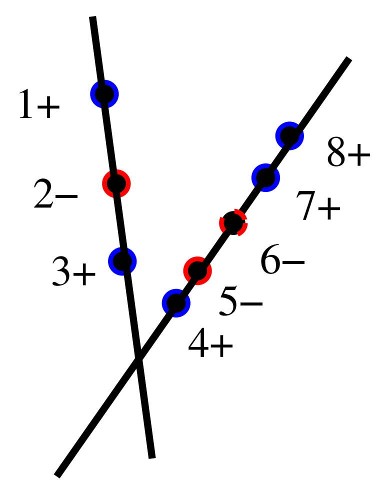
MHV





NMHV



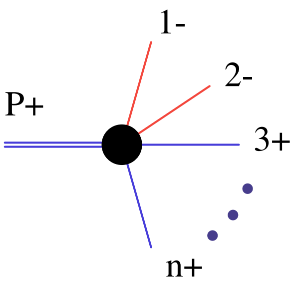


MHV rules

Start from MHV amplitude and define off-shell vertices

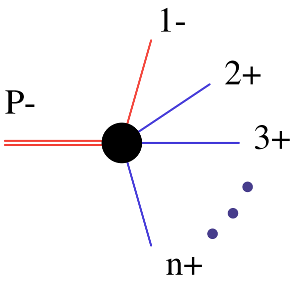
Cachazo, Svrcek and Witten

$$V(1^-,2^-,3^+,\ldots,n^+,P^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\cdots\langle n-1n\rangle\langle nP\rangle\langle P1\rangle}$$



and

$$V(1^-,2^+,3^+,\ldots,n^+,P^-) = \frac{\langle 1P\rangle^4}{\langle 12\rangle\cdots\langle n-1n\rangle\langle nP\rangle\langle P1\rangle}$$



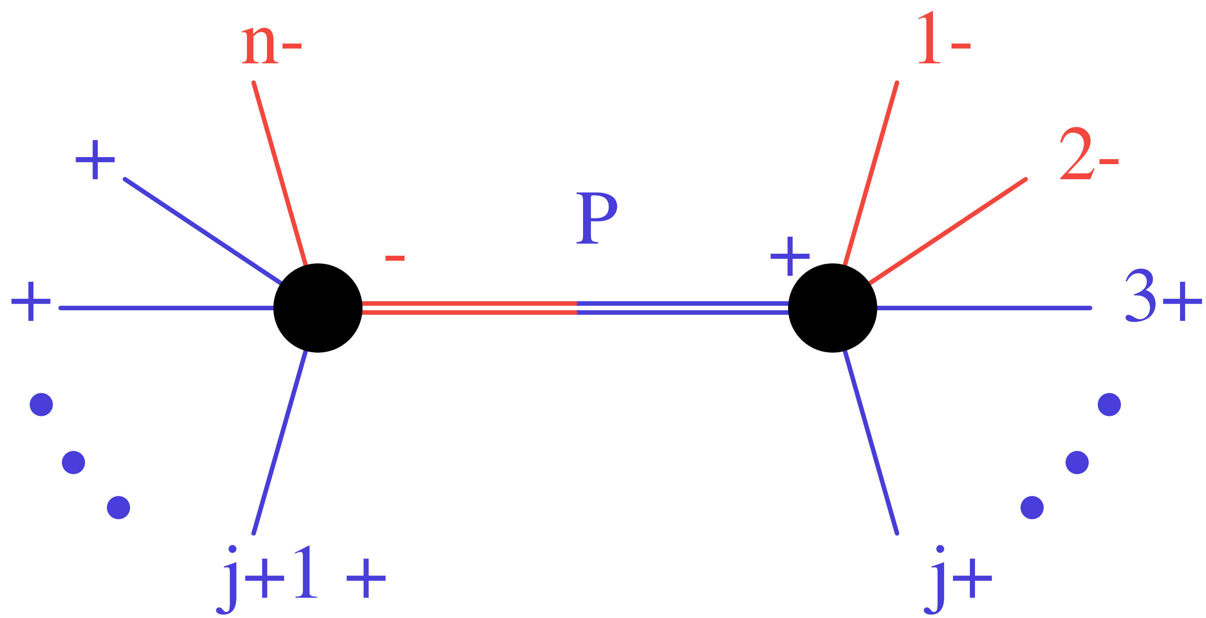
Crucial step is off-shell continuation $P^2 \neq 0$:

$$\langle iP\rangle = \frac{\langle i^-|\not{P}|\eta^-\rangle}{[P\eta]} = \sum_j \frac{\langle i^-|\not{j}|\eta^-\rangle}{[P\eta]}$$

where $P = \sum_j j$ and η is lightlike auxiliary vector

MHV rules

Must connect up a positive helicity off-shell line with a negative helicity off-shell line



Connecting two MHV's $\Rightarrow$ amplitude with 3 negative helicities
Connecting three MHV's $\Rightarrow$ amplitude with 4 negative helicities
etc.

Example: six gluon scattering

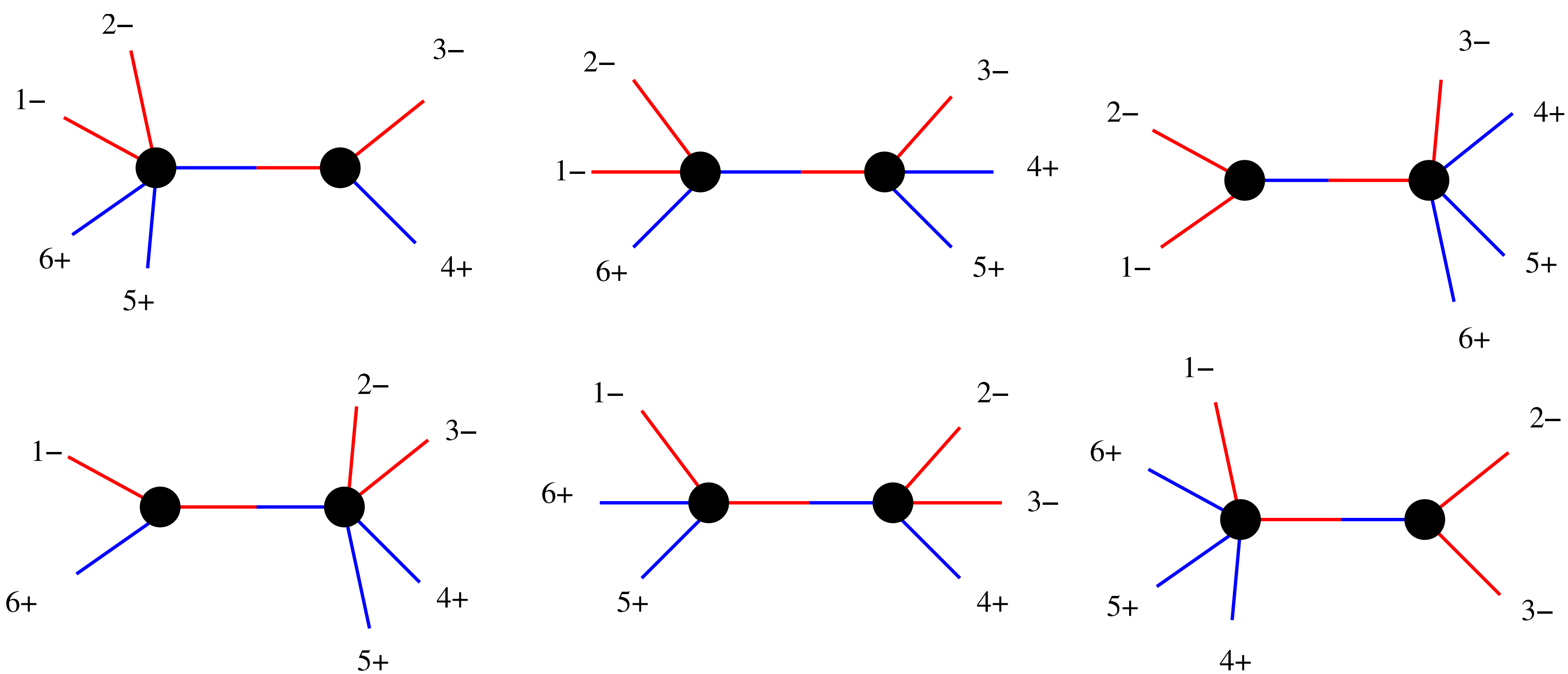
As an example, lets use the MHV rules to calculate one of the first non-MHV amplitudes

$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$$

Step 1 Draw all the allowed MHV diagrams

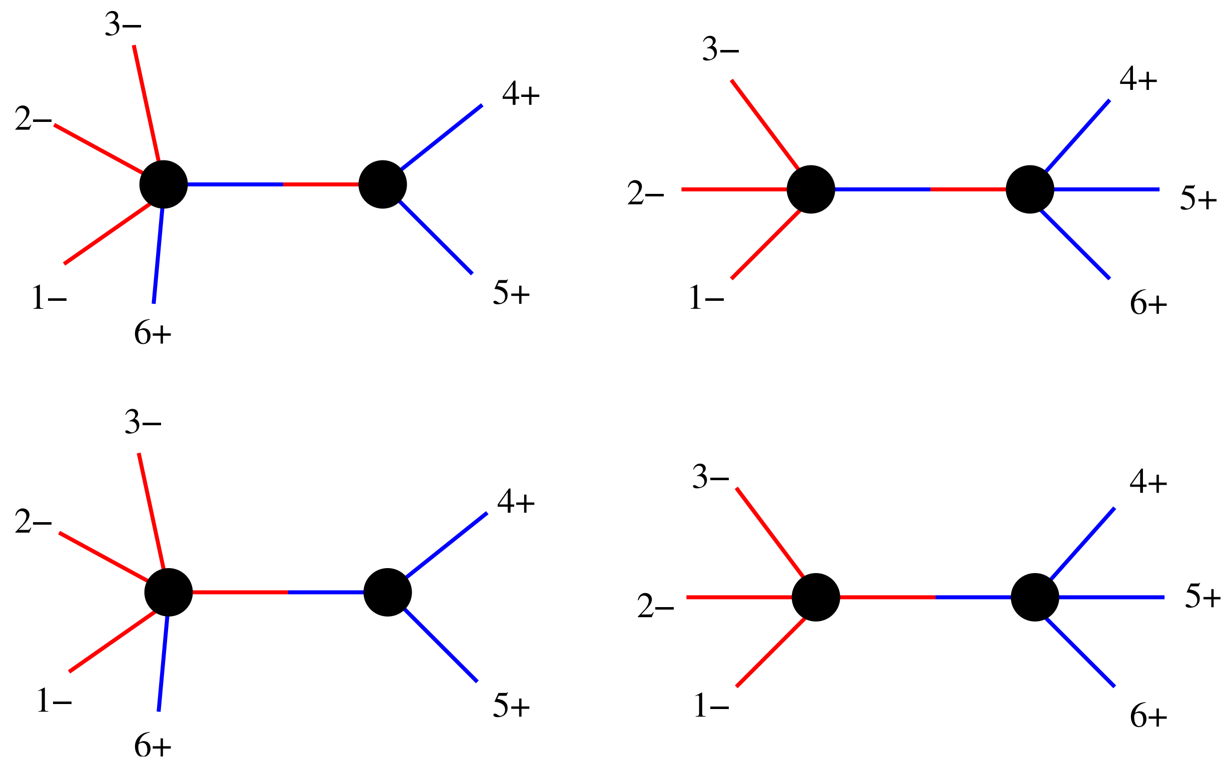
Example: six gluon scattering

There are six MHV graphs



Example: six gluon scattering

Some graphs are not allowed e.g.



Example: six gluon scattering

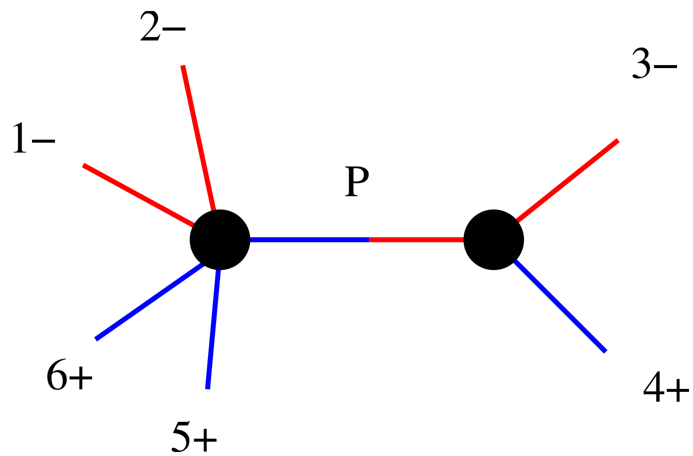
As an example, lets use the MHV rules to calculate one of the first non-MHV amplitudes

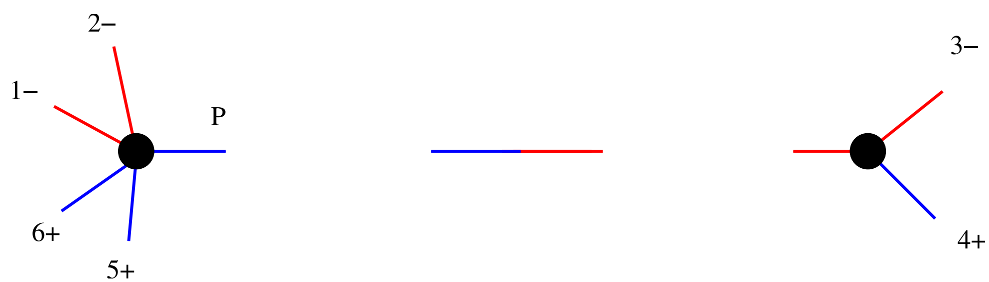
$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$$

Step 1 Draw all the allowed MHV diagrams

Step 2 Apply MHV rules to each diagram

Example: six gluon scattering: diagram 1

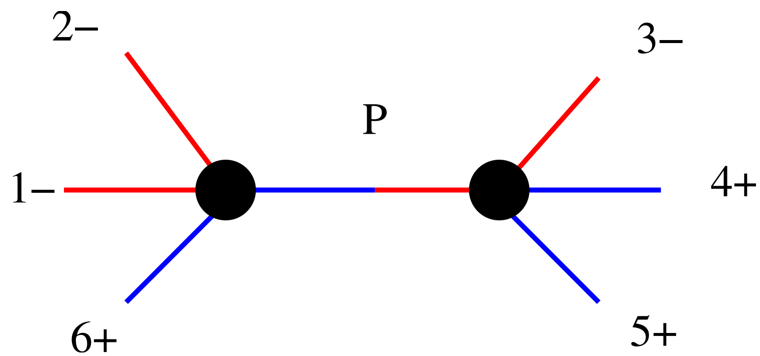


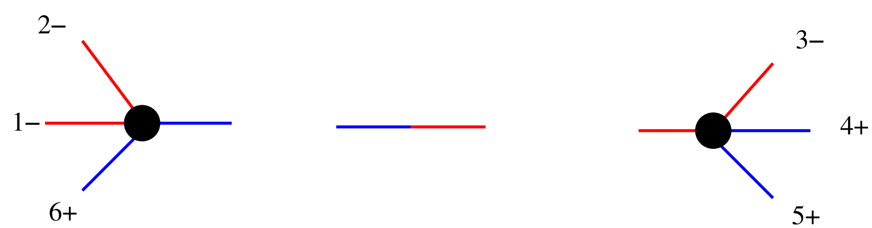


$$\frac{\langle 12\rangle^4}{\langle 56\rangle\langle 61\rangle\langle 12\rangle\langle 2|P|\eta\rangle\langle 5|P|\eta\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|P|\eta\rangle^4}{\langle 34\rangle\langle 4|P|\eta\rangle\langle 3|P|\eta\rangle}$$

with $P = 3 + 4 = -(1 + 2 + 5 + 6)$

Example: six gluon scattering: diagram 2





$$\frac{\langle 12\rangle^4}{\langle 61\rangle\langle 12\rangle\langle 2|P|\eta\rangle\langle 6|P|\eta\rangle} \times \frac{1}{s_{345}} \times \frac{\langle 3|P|\eta\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 5|P|\eta\rangle\langle 3|P|\eta\rangle}$$

with $P = 3 + 4 + 5 = -(1 + 2 + 6)$

Example: six gluon scattering

As an example, lets use the MHV rules to calculate one of the first non-MHV amplitudes

$$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$$

Step 1 Draw all the allowed MHV diagrams

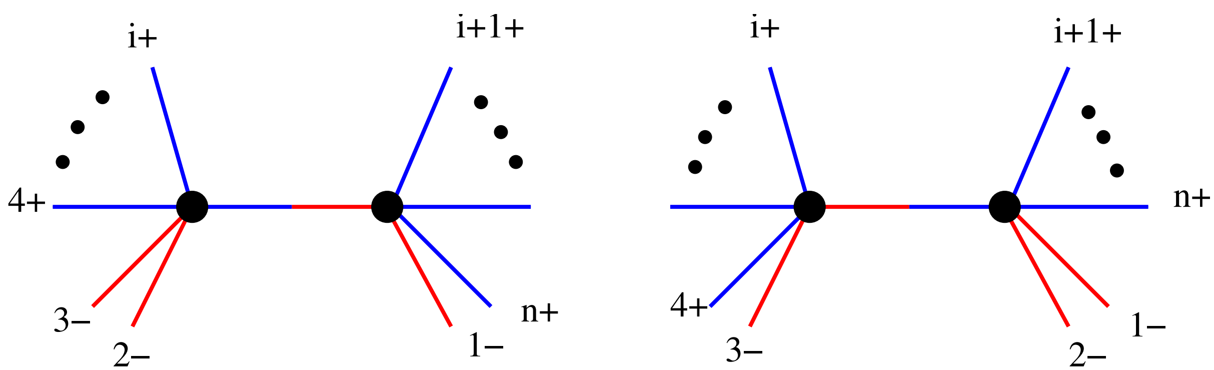
Step 2 Apply MHV rules to each diagram

Step 3 Add up diagrams and check η independence

Next-to MHV amplitude for n gluons

Simplest case: $A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ $\qquad$ $2(n-3)$ **graphs**

Cachazo, Svrcek and Witten



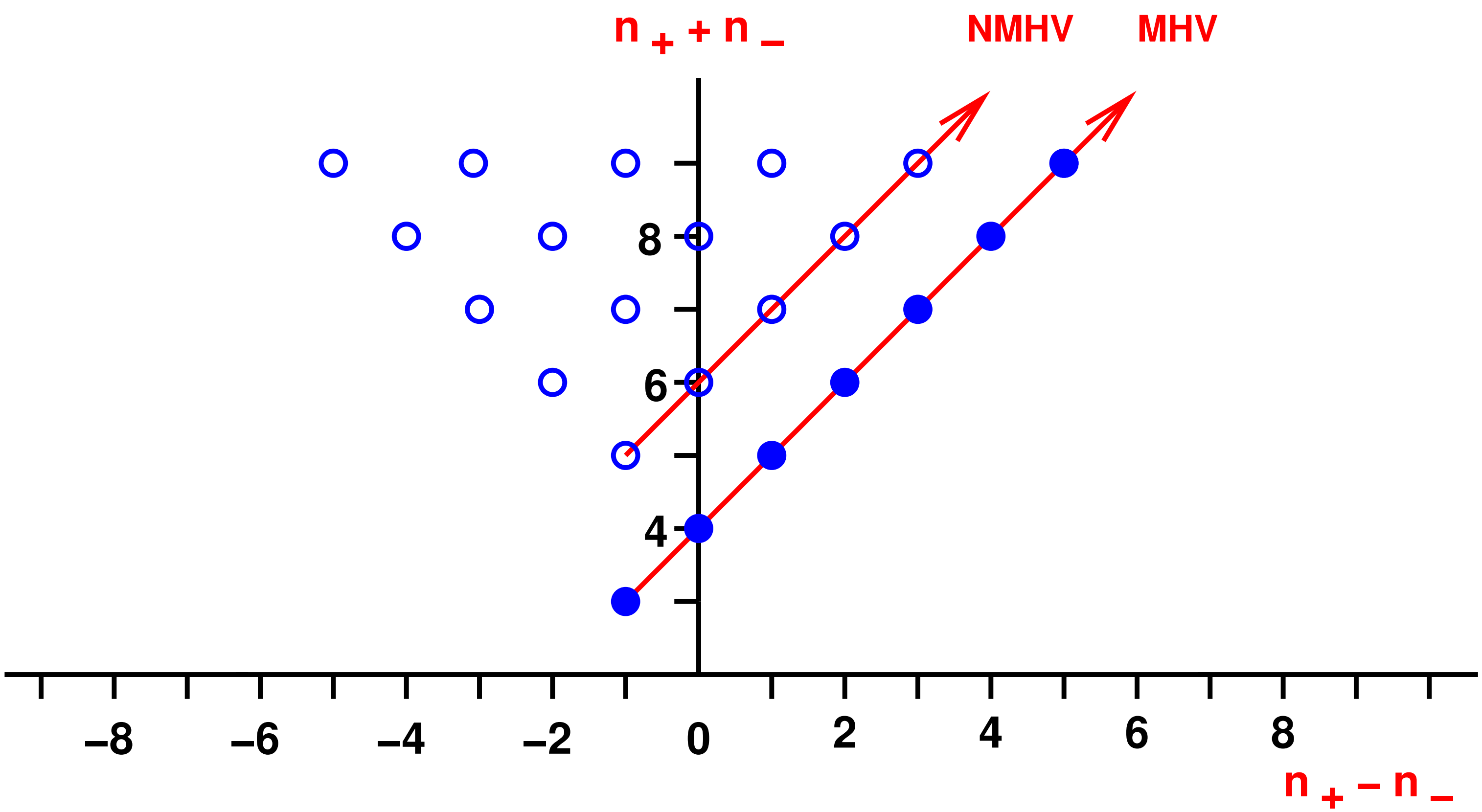
$$A = \sum_{i=3}^{n-1} \frac{\langle 1(2,i)\rangle^3}{\langle (2,i) i+1\rangle\langle i+1 i+2\rangle \ldots \langle n1\rangle} \frac{1}{s_{2,i}^2} \frac{\langle 23\rangle^3}{\langle (2,i)2\rangle\langle 34\rangle \cdots \langle i(2,i)\rangle}$$
$$+ \sum_{i=4}^{n} \frac{\langle 12\rangle^3}{\langle 2(3,i)\rangle\langle (3,i)i+1\rangle \ldots \langle n1\rangle} \frac{1}{s_{3,i}^2} \frac{\langle (3,i)3\rangle^3}{\langle 34\rangle \cdots \langle i-1i\rangle\rangle\langle i(3,i)\rangle}.$$

where $(k,i) = k + \cdots + i$ **and the off-shell continuation is suppressed**

$\Rightarrow$ **Lorentz invariant and gauge invariant expressions**

Generating all the tree amplitudes

Amplitudes with $i-$ and $j+$ helicities



- MHV rules always adds one negative helicity and any number of positive helicities
  $\Rightarrow$ maps out all allowed tree amplitudes

Other processes

MHV rules have been generalised to many other processes

- √ with massless fermions - quarks, gluinos

  Georgiou and Khoze; Wu and Zhu; Georgiou, EWNG and Khoze

- √ with massless scalars - squarks

  Georgiou, EWNG and Khoze; Khoze

- √ with an external Higgs boson

  Dixon, EWNG, Khoze; Badger, EWNG, Khoze

- √ with an external weak boson

  Bern, Forde, Kosower and Mastrolia

Has provided **new** results for n-particle amplitudes
Also useful for studying infrared properties of amplitudes

Birthwright, EWNG, Khoze and Marquard

Processes with fermions

Similar colour decomposition

$$\mathcal{A}_n(1,\ldots,\Lambda_r,\Lambda_s,\ldots,n) = \sum_{perms} (T^{a_1}\ldots T^{a_n})_{r,s} A_n(\Lambda_r,1,\ldots,n,\Lambda_s)$$

MHV amplitude with 2 fermions and $n-2$ gluons

$$A_n(g_t^-,\Lambda_r^-,\Lambda_s^+) = \frac{\langle tr\rangle^3\langle ts\rangle}{\prod_{i=1}^n \langle i\ i+1\rangle}$$

MHV amplitude with 4 fermions and $n-4$ gluons

$$A_n(\Lambda_r^-,\Lambda_s^+,\Lambda_t^-,\Lambda_u^+) = \frac{\langle rt\rangle^3\langle su\rangle}{\prod_{i=1}^n \langle i\ i+1\rangle}$$

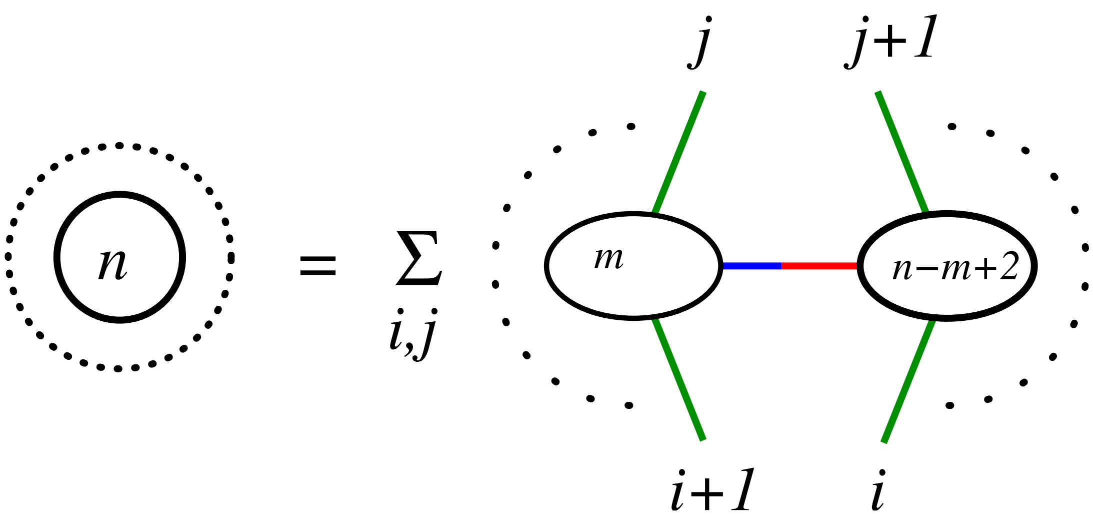
$\Rightarrow$ similar scalar graph construction for fermionic amplitudes

Georgiou and Khoze; Wu and Zhu; Georgiou, EWNG and Khoze

Recursive MHV amplitudes

As the number of negative helicity legs grows, the number of MHV diagrams grows
$\Rightarrow$ Use previously computed on-shell NMHV amplitudes as building blocks for recursion relation

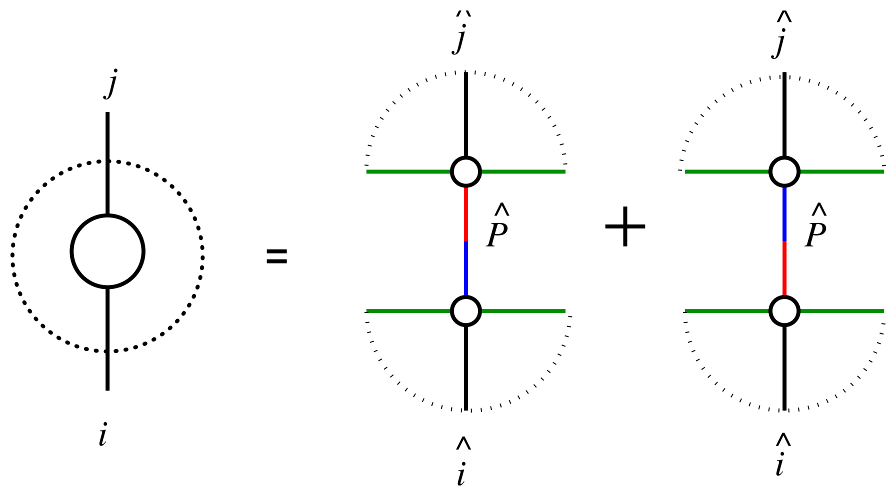
Bena, Bern and Kosower



connected by same off-shell continuation as before.
Each blob is an amplitude with fewer particles and fewer negative helicities.
$\Rightarrow$ easily programmed

BCF recursion relations

Based on experience with one-loop amplitudes, Britto, Cachazo and Feng proposed a new set of **on-shell** recursion relations



Britto, Cachazo and Feng
Britto, Cachazo, Feng and Witten

hatted momenta are shifted to put on-shell

$$\hat{i} = i + z\eta, \qquad \hat{j} = j - z\eta, \qquad \hat{P} = P + z\eta$$

$\Rightarrow$ each vertex is an **on-shell** amplitude

BCF recursion relations

- It turns out that the shift η is not a momentum, but

$$\eta = \lambda_i \tilde{\lambda}_j \qquad OR \qquad \eta = \lambda_j \tilde{\lambda}_i$$

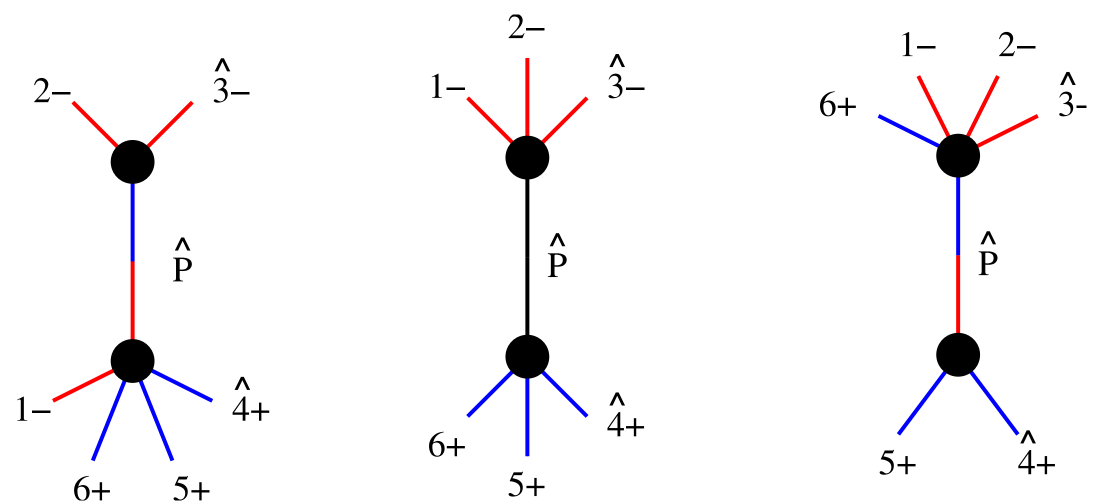
- The parameter z is given by

$$z = \frac{P^2}{\langle j P i]}$$

- Easy to prove that recursion relation is valid using complex analysis
- Requires on-shell three-point vertex contributions - both MHV and $\overline{\mathrm{MHV}}$.

BCF - six gluon example

If we select 3 and 4 to be the special gluons, there are only three diagrams (for any helicities)



For this helicity assignment, the middle one is zero!.

$A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

$$= \frac{1}{\langle 5|\not{3}+\not{4}|2\rangle}\left(\frac{\langle 1|\not{2}+\not{3}|4\rangle^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{234}} + \frac{\langle 3|\not{4}+\not{5}|6\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle s_{345}}\right)$$

Extremely compact (and correct) results for up to 8 gluons

Other processes

BCF recursion relations have been generalised to other processes

- √ with massless fermions - quarks, gluinos

  Luo and Wen

- √ gravitons

  Bedford, Brandhuber, Spence and Travaglini; Cachazo and Svrcek

There is nothing (in principle) to stop this approach being applied to particles with mass.

One loop amplitudes

- So far, supersymmetry was not a major factor - tree level amplitudes same for $\mathcal{N}=4$ $\mathcal{N}=1$ and QCD
- Not true at the loop level due to circulating states

$$
\begin{aligned}
A_n^{\mathcal{N}=4} &= A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]} \\
A_n^{\mathcal{N}=1,chiral} &= A_n^{[1/2]} + A_n^{[0]} \\
A_n^{glue} &= A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1,chiral} + A_n^{[0]}
\end{aligned}
$$

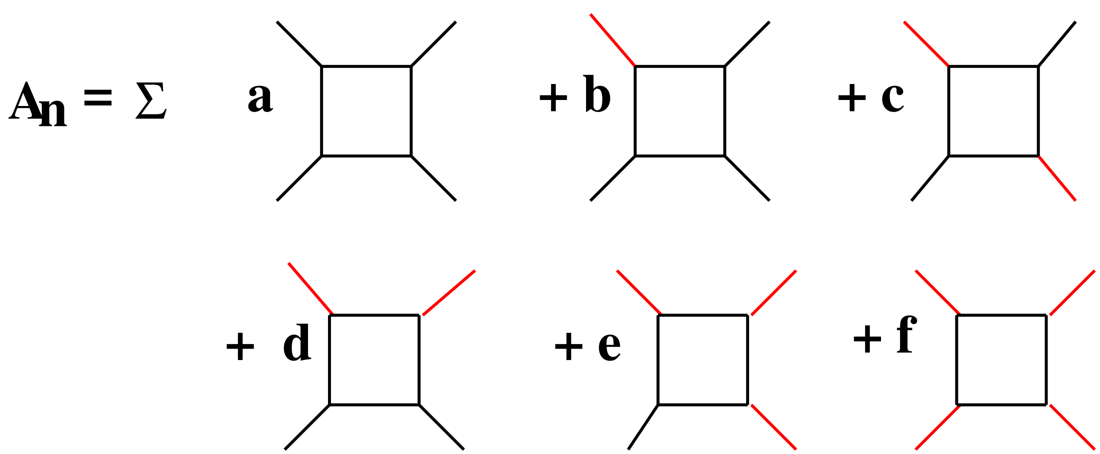
- All plus and nearly all-plus amplitudes do not vanish for non-supersymmetric QCD
- A lot of progress by a lot of people

SUSY QCD loops

- √ $\mathcal{N}=4$ and $\mathcal{N}=1$ one-loop amplitudes are constructible from their 4-dimensional cuts
  $\Rightarrow$ employ unitarity techniques

  Bern, Dixon, Dunbar, Kosower

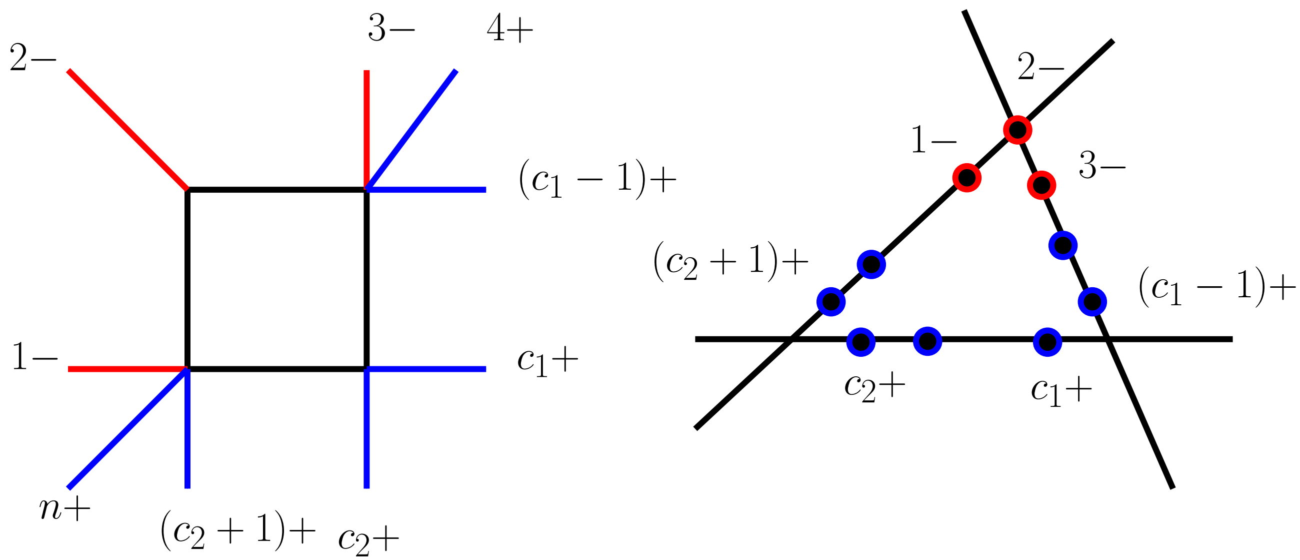
- √ For $\mathcal{N}=4$ **all** amplitudes are a linear combination of known box integrals



Twistor space interpretation

- **Coefficients of boxes have very interesting structures.**

  Bern, Del Duca, Dixon, Kosower; Britto, Cachazo, Feng

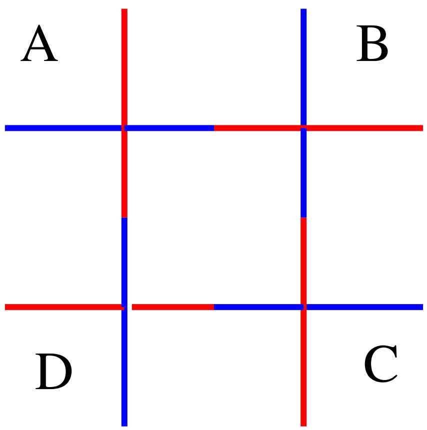


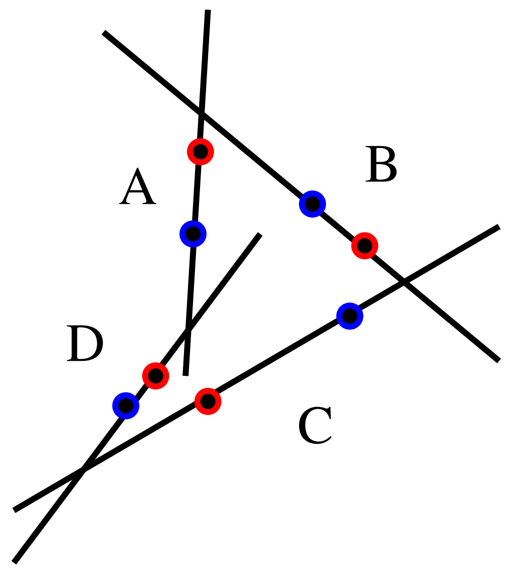
Twistor space interpretation

- Four mass box first appears in eight-point amplitude with four negative and four positive helicities

Bern, Dixon, Kosower

e.g.





QCD loops

QCD amplitudes more complicated

(a) Not 4-dimensional cut constructible. Rational function contribution not probed by 4-d cut

(b) All plus and almost all plus amplitudes not zero - but rational functions. Not protected by SWI.

Nevertheless, all four-point and five-point amplitudes known:
Recent progress

√ On-shell recurrence relations for all plus and almost all plus amplitudes

Bern, Dixon and Kosower

Recursion relations complicated by double pole terms and boundary terms

√ Scalar six-point NMHV amplitudes

Bidder, Bjerrum-Bohr, Dunbar and Perkins

Computed parts of six-point QCD amplitudes that are obtainable using 4-dimensional cut constructibility

Summary - New rules for tree-level amplitudes

- MHV rules — **Cachazo, Svrcek and Witten**
  - √ New way of computing amplitudes with gluons and massless quarks
  - √ Higgs coupling to massless quarks and gluons

    Dixon, EWNG, Khoze; Badger, EWNG, Khoze
  - √ Vector bosons coupling to massless quarks

    Bern, Forde, Kosower and Mastrolia
- BCF recursion relations — **Britto, Cachazo and Feng; Britto, Cachazo, Feng and Witten**
  - √ Extended to quarks

    Luo and Wen
  - √ and gravitons

    Bedford, Brandhuber, Travaglini, Spence; Cachazo, Svrcek

Summary - Progress for one-loop amplitudes

√ $\mathcal{N} = 4$ amplitudes
almost at the point where coefficients of boxes can be read off - using quadruple cuts and holomorphic anomaly

Britto, Cachazo and Feng

⇒ All NMHV amplitudes

Bern, Dixon and Kosower

√ $\mathcal{N} = 1$ MHV amplitudes and 6-point NMHV amplitudes

√ Application to one-loop gravity

Bern, Bjerrum-Bohr, Dunbar

? QCD amplitudes

Bedford, Brandhuber, Spence and Travaglini; Bern, Dixon and Kosower; Bidder, Bjerrum-Bohr, Dunbar and Perkins

A very exciting and rapidly developing field
Expect more important results soon