Construction of a relativistic field theory

Lagrangian $$L = T - V$$ (Nonrelativistic mechanics)

Action $$S = \int_{t_1}^{t_2} L \, dt$$

- Classical path … minimises action

Feynman lectures

- Quantum mechanics … sum over all paths with amplitude $\propto e^{iS/\hbar}$

Lagrangian invariant under all the symmetries of nature

-makes it easy to construct viable theories

Lagrangian formulation of the Klein Gordon equation

$$L = \int \mathcal{L}\; d^3x, \quad \mathcal{L} \quad \text{lagrangian density}$$

Klein Gordon field $\phi(x)$

$$\mathcal{L} = \underbrace{\left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x)}_{T} - \underbrace{m^2 \phi(x)^\dagger \phi(x)}_{V}$$

Manifestly Lorentz invariant

Classical path :

Dimensions?

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = 0$$

Euler Lagrange equation

Euler Lagrange equs

$$S=\int_{t_1}^{t_2} L\,dt=\int \mathrm{L}\,(\phi,\partial_\mu\phi)\,d^4x$$

Principle of least action :

0 (surface integral)

$$0=\delta S=\int d^4x\left\{\frac{\partial \mathrm{L}}{\partial\phi}\delta\phi+\frac{\partial \mathrm{L}}{\partial(\partial^\mu\phi)}\delta(\partial^\mu\phi)\right\}$$

$$=\int d^4x\left\{\frac{\partial \mathrm{L}}{\partial\phi}\delta\phi-\partial_\mu\left(\frac{\partial \mathrm{L}}{\partial(\partial^\mu\phi)}\right)\delta\phi+\partial_\mu\left(\frac{\partial \mathrm{L}}{\partial(\partial^\mu\phi)}\delta\phi\right)\right\}$$

⇒

$$\boxed{\frac{\partial \mathrm{L}}{\partial\phi}-\partial^\mu\frac{\partial \mathrm{L}}{\partial(\partial^\mu\phi)}=0}$$

Euler Lagrange equation

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Manifestly Lorentz invariant

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = 0$$

Euler Lagrange equation

⇨ $$\boxed{(\partial_\mu \partial^\mu + m^2)\phi = 0}$$

Klein Gordon equation

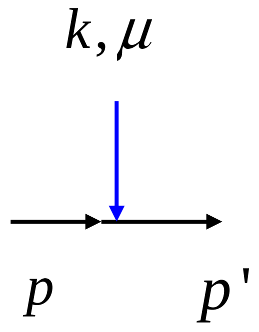
The Lagrangian and Feynman rules

Associate with the various terms in the Lagrangian a set of propagators and vertex factors

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.
- The remaining terms in the Lagrangian are associated with interaction vertices. The Feynman vertex factor is just given by the coefficient of the corresponding term in $i\mathcal{L}$

e.g. $\left((\partial_\mu - ieA_\mu)\phi(x)\right)^\dagger (\partial^\mu - ieA^\mu)\phi(x) \quad \rightarrow \quad -ieA_\mu(\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi)$

$$i\mathcal{L} = -ie\underbrace{(p_i + p_f)}A_\mu\phi^*\phi$$



Fundamental experimental objects

$\Gamma(a_1 \to b_1 b_2 \ldots b_n)$ **Decay width = 1/lifetime** (Dimension 1/T=M)

$\sigma(a_1 a_2 \to b_1 b_2 \ldots b_n)$ **Cross section** (Dimension $L^2=M^{-2}$)

Units “barn”

$$\begin{aligned}
&1\,barn = 10^2\ fm^2 \\
&1\,mb \quad = 10^{-1}\ fm^2 \quad \text{"}milli\text{"} \\
&1\,\mu b \quad = 10^{-4}\ fm^2 \quad \text{"}micro\text{"} \\
&1\,nb \quad = 10^{-7}\ fm^2 \quad \text{"}nano\text{"} \\
&1\ pb \quad = 10^{-10}\ fm^2 \quad \text{"}\ pico\text{"} \\
&1\ fb \quad = 10^{-13}\ fm^2 \quad \text{"}\ fempto\text{"}
\end{aligned}$$

(Natural Units $1GeV^{-2} = 0.39mb$)

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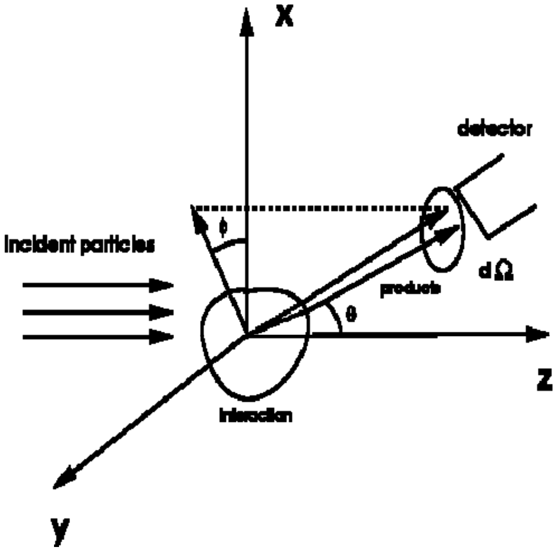

Cross section = $\dfrac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$

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Momenta of final state forms phase space

$$\boxed{\text{Cross section} = \frac{\text{Transition rate x Number of final states}}{\text{Initial flux}}}$$

For a single particle the number of final states in volume V with momenta

in element d^3p is $\frac{Vd^3p}{(2\pi)^3}$ $\Longrightarrow$ $\boxed{\Pi_{i=1}^{n} \frac{Vd^3p}{(2\pi)^3}}$

Fundamental experimental objects

$\Gamma(a_1 \to b_1 b_2 \ldots b_n)$ **Decay width = 1/lifetime**

$\sigma(a_1 a_2 \to b_1 b_2 \ldots b_n)$ **Cross section**

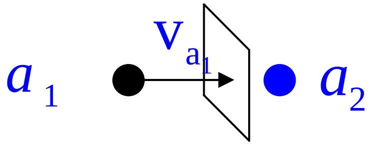
$$\boxed{\Pi_{i=1}^{n} \frac{Vd^3 p}{(2\pi)^3}}$$

$$\text{Cross section} = \frac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$$

$$\boxed{\frac{|\mathrm{v}_{a_1}|}{V} \times \frac{1}{V}}$$

particles passing through unit area in unit time

target particles per unit volume



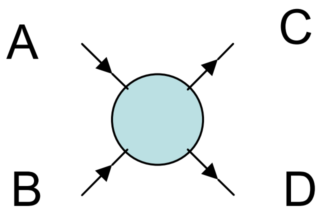
(Lab frame)

The transition rate

$$T_{fi} = -\int d^4x\, \phi_f^*(x) V(x) \phi_i(x) + \dots$$

$$\boxed{\phi_{i,f} \to f_p^{\pm} = e^{\mp ip.x}\frac{1}{\sqrt{2p^0V}} \equiv \frac{N}{\sqrt{V}} e^{\mp ip.x}}$$

e.g.



Transition rate per unit volume

$$W_{fi} = \frac{|T_{fi}|^2}{TV}$$

$$\boxed{\phi_{f,i} = e^{\mp ip.x}}$$

$$T_{fi} = -\frac{N_A N_B N_C N_D}{V^2}(2\pi)^4 \delta^4(p_C + p_D - p_A - p_B)\mathrm{M}_{\mathrm{fi}}$$

$$\boxed{W_{fi} = (2\pi)^4 \frac{\delta^4(p_C + p_D - p_A - p_B)|\mathrm{M}|^2}{V^4}\left(\frac{1}{2E_A}\right)\left(\frac{1}{2E_B}\right)\left(\frac{1}{2E_C}\right)\left(\frac{1}{2E_D}\right)}$$

The cross section

Cross section = $\dfrac{\text{Transition rate x Number of final states}}{\text{Initial flux}}$

$$d\sigma = \frac{V^2}{|\mathbf{v_A}| 2E_A 2E_B} \frac{1}{V^4} |\mathrm{M}|^2 \frac{(2\pi)^4}{(2\pi)^6} \delta^4(p_C + p_D - p_A - p_B) \frac{d^3 p_C}{2E_C} \frac{d^3 p_D}{2E_D} V^2$$

$$\boxed{d\sigma = \frac{|\mathrm{M}|^2}{F} dQ}$$

$$dQ = (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) \frac{d^3 p_C}{(2\pi)^3 2E_C} \frac{d^3 p_D}{(2\pi)^3 2E_D}$$

Lorentz
Invariant
Phase
space

$$F = |\mathbf{v_A}| 2E_A 2E_B$$
$$= 4((p_A . p_B)^2 - m_A^2 m_B^2)^{1/2}$$

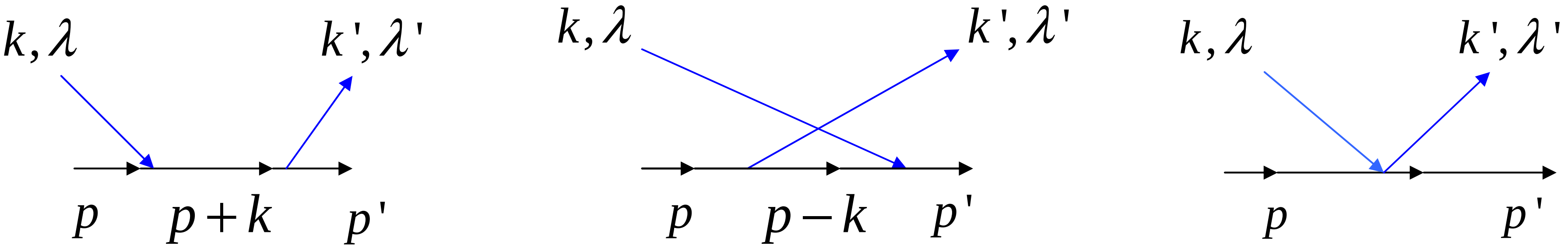
The decay rate

$$\boxed{d\Gamma = \frac{1}{2E_A}|\mathrm{M}|^2\, dQ}$$

$$dQ = (2\pi)^4 \delta^4(p_A - p_{B_1} \ldots - p_{B_n}) \frac{d^3 p_{B_1}}{(2\pi)^3 2E_{B_1}} \cdots \frac{d^3 p_{B_n}}{(2\pi)^3 2E_{B_n}}$$

Compton scattering of a π meson

$$\gamma\pi \to \gamma\pi$$

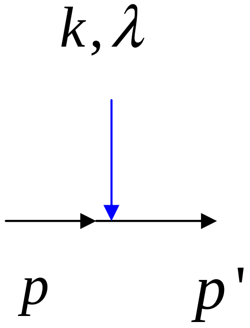


Feynman rules

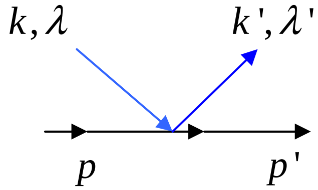
Klein Gordon

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi$$

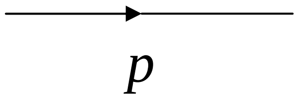
$$V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$



$$-ie(p_\lambda + p'_\lambda)$$



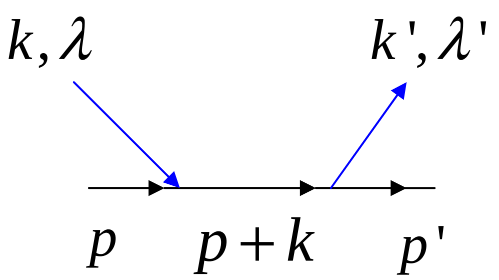
$$ie^2$$

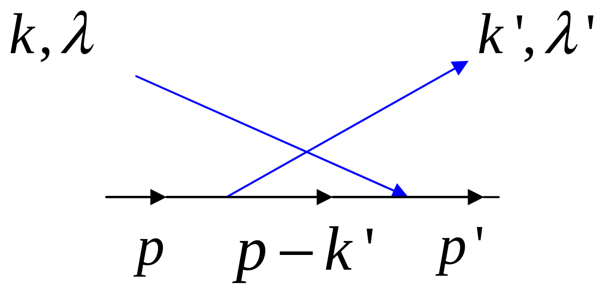


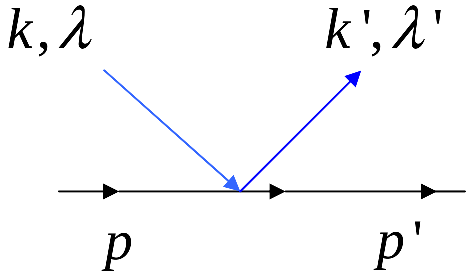
$$\frac{i}{p^2 - m^2}$$

External photon ε^λ

Compton scattering of a π meson





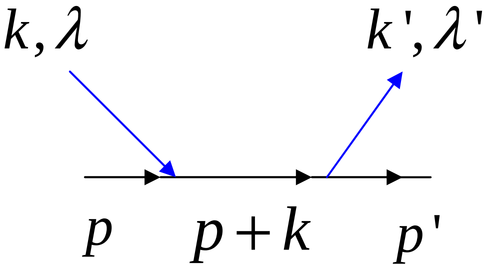


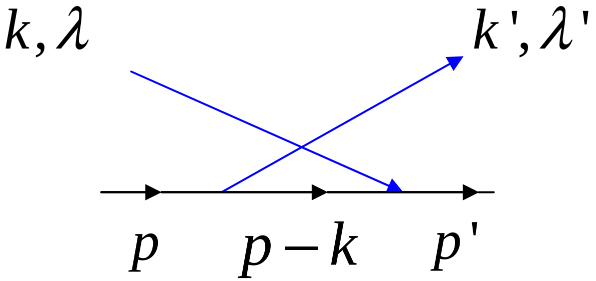
$$\mathrm{iM}_{fi} = (-ie)^2[\varepsilon.(2p+k)\frac{i}{(p+k)^2-m^2}\varepsilon'.(2p'+k')$$

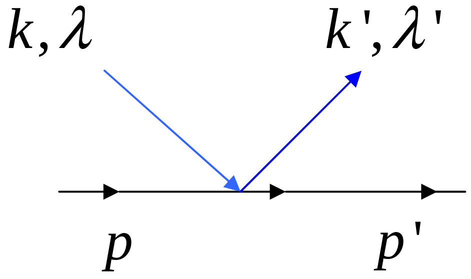
$$+\varepsilon.(2p'-k)\frac{i}{(p-k')^2-m^2}\varepsilon'.(2p-k')-2i\varepsilon.\varepsilon']$$

$$d\sigma = \frac{V^2}{|\mathbf{v_A}|2E_A 2E_B}\frac{1}{V^4}|\mathrm{M}|^2\frac{(2\pi)^4}{(2\pi)^6}\delta^4(p_C+p_D-p_A-p_B)\frac{d^3p_C}{2E_C}\frac{d^3p_D}{2E_D}V^2$$

Compton scattering of a π meson







$$\mathrm{M}_{fi} = \varepsilon.(2p+k)\frac{i}{(p+k)^2-m^2}\varepsilon'.(2p'+k')$$

$$+\varepsilon.(2p'-k')\frac{i}{(p+k)^2-m^2}\varepsilon'.(2p'-k')-2i\varepsilon.\varepsilon'$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\alpha^2}{m^2}\frac{(\varepsilon.\varepsilon')^2}{[1+\frac{k}{m}(1-\cos\theta)]^2}}$$

$(\varepsilon.p=\varepsilon'.p=0 \text{ gauge})$

$$\sigma_{total}\,|_{k=0} = \int\frac{d\sigma}{d\Omega}.d\Omega = \frac{8\pi\alpha^2}{3m_\pi^2} \Box\ 8.10^{-2}GeV^{-2} = 3.10^{-2}mb$$

$$\sigma_{total}\,|_{k/m>>1} \Box\ \frac{2\pi\alpha^2}{mk}$$