

The inclusion of fermions – J=1/2 particles

Weyl spinors

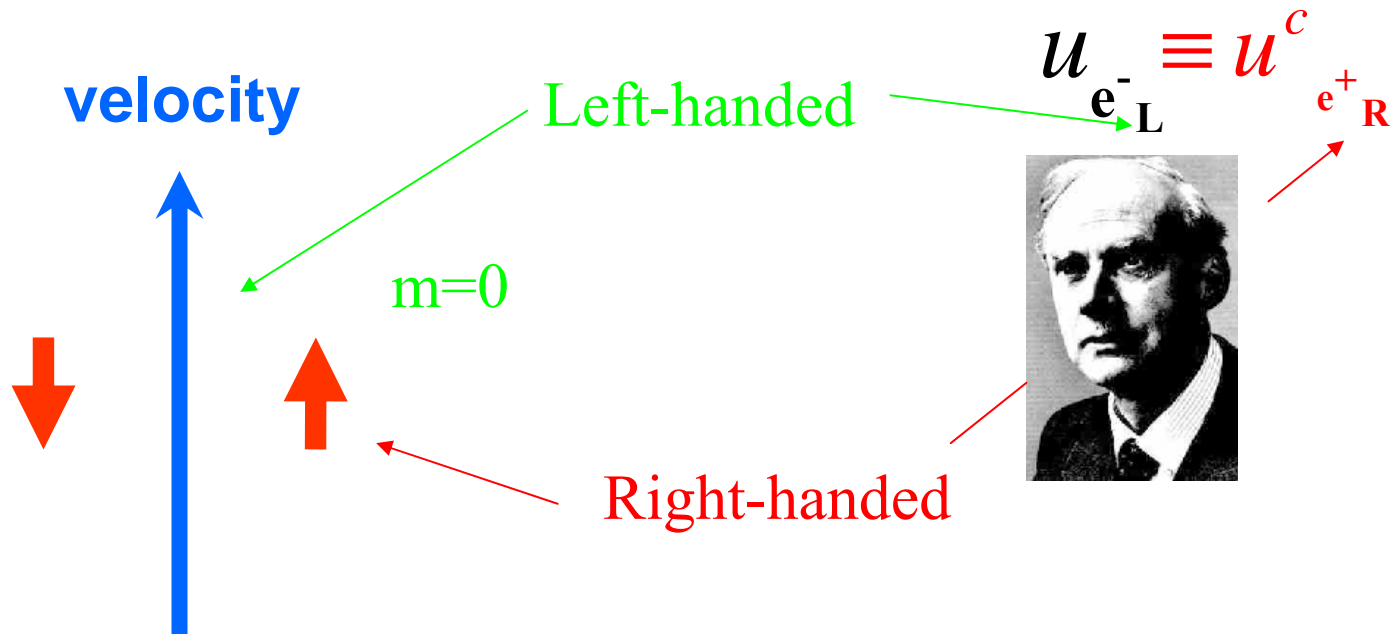
$$\left(\frac{1}{2}, 0\right) \quad \left(0, \frac{1}{2}\right)$$

$$\psi_L \quad \psi_R$$

Dirac spinor

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

2-component spinors of SU(2) $\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$



The spinor structure follows from the representation structure of the Lorentz Group ...

- The Lorentz transformations form a group, G ($g_1 g_2 \in G$ if $g_1, g_2 \in G$)

Rotations

$$R(\theta) = e^{-i\mathbf{J}\cdot\theta/\hbar}, \quad J_z = i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}).. \quad \text{Angular momentum operator}$$

(c.f. $\mathbf{J} = \mathbf{r} \times \mathbf{p}$)

$$[J_i, J_j] = i\hbar \sum_{k=1}^3 \varepsilon_{ijk} J_k$$

$SO(3)$ ($SU(2)$)

e.g. Spin $R(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\sigma\cdot\theta/2\hbar} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Lorentz group

Rotations J_i Boosts K_i

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

}

Generate the group SO(3,1)

$$(M_{\rho\sigma} = i(x_\rho \frac{\partial}{\partial x^\sigma} - x_\sigma \frac{\partial}{\partial x^\rho}) \quad J_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \quad K_i = M_{0i})$$

To construct representations a more convenient (non-Hermitian) basis is

$$N_i = \frac{1}{2} (J_i + iK_i)$$

$$[N_i, N_j] = i\epsilon_{ijk} N_k$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger$$

$$[N_i, N_j^\dagger] = 0$$

}

$SU(2) \otimes SU(2)$ representation (n, m)

The Lorentz group

Rotations J_i Boosts K_i

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$$N_i = \frac{1}{2} (J_i + iK_i)$$

Representations $J_i = N_i + N_i^\dagger$

$$(n, m) \quad J = n + m$$

$$[N_i, N_j] = i\epsilon_{ijk} N_k$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger$$

$$[N_i, N_j^\dagger] = 0$$

$$(0, 0) \quad \text{scalar} \quad J=0$$

$$(\frac{1}{2}, 0), (0, \frac{1}{2}) \quad \text{LH and RH spinors} \quad J=\frac{1}{2}$$

$$(\frac{1}{2}, \frac{1}{2}) \quad \text{vector} \quad J=1, \quad \text{etc}$$

Weyl spinors

$$\begin{matrix} (\frac{1}{2}, 0) & (0, \frac{1}{2}) \\ \psi_L & \psi_R \end{matrix}$$

2-component spinors of SU(2)

Rotations and Boosts

$$\psi_{L(R)} \rightarrow S_{L(R)} \psi_{L(R)}$$

$$\begin{aligned} S_{L(R)} &= e^{i\frac{\sigma}{2} \cdot \omega} : \text{Rotations} \\ S_{L(R)} &= e^{\pm\frac{\sigma}{2} \cdot \mathbf{v}} : \text{Boosts} \end{aligned}$$

Dirac spinor

Can combine ψ_L, ψ_R to form a 4-component "Dirac" spinor

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

Lorentz transformations

$$\psi \rightarrow e^{i\omega\sigma} \psi, \quad \omega\sigma = \omega^{\mu\nu} \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \omega^{\mu\nu}$$

where

$$\gamma_0 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Weyl basis

$$\omega^{0i} \rightarrow \text{boosts}, \quad \omega^{ij} \rightarrow \text{rotations} \quad i, j = 1, 2, 3$$

Weyl spinors

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(Dirac gamma matrices, ...new 4-vector γ_μ)

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

Note : $\psi_{L(R)} = \frac{1}{2} (1 \mp \gamma_5) \psi$

The Dirac equation

Fermions described by 4-cpt Dirac spinors ψ

- $\psi^\dagger \gamma^0 \psi \equiv \bar{\psi} \psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$ Lorentz invariant
- New 4-vector γ_μ

The Lagrangian

$$\mathcal{L} = i \bar{\psi} \gamma_\mu \partial^\mu \psi - m \bar{\psi} \psi \quad \text{Dimension?}$$

From Euler Lagrange equation obtain the Dirac equation

$$(i\gamma_\mu \partial^\mu - m)\psi = 0$$

U(1) symmetry

$$\psi \rightarrow e^{i\alpha} \psi \quad \rightarrow \quad j^\mu = -e \bar{\psi} \gamma^\mu \psi$$

Feynman rules

$$\begin{array}{c} p \\ \longrightarrow \end{array} \quad \frac{i}{(p - m)}$$

$$\begin{array}{c} \mu \\ \downarrow \\ \longrightarrow \end{array} \quad ie\gamma^\mu$$

Fermion masses

$$\psi^\dagger \gamma^0 \psi \equiv \bar{\psi} \psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$$

Lorentz invariant

$$L = m(\bar{e}_L e_R + \bar{e}_R e_L)$$

Dirac mass

$$\sigma_2 \psi_R^\dagger \equiv \psi_L$$

$$(\psi_L^T \sigma_2 \psi_L + hc)$$

Lorentz invariant

Only term allowed by charge conservation is

$$L' = m(\nu_L^T \sigma_2 \nu_L + hc)$$

Majorana mass

Fermions and the weak interactions

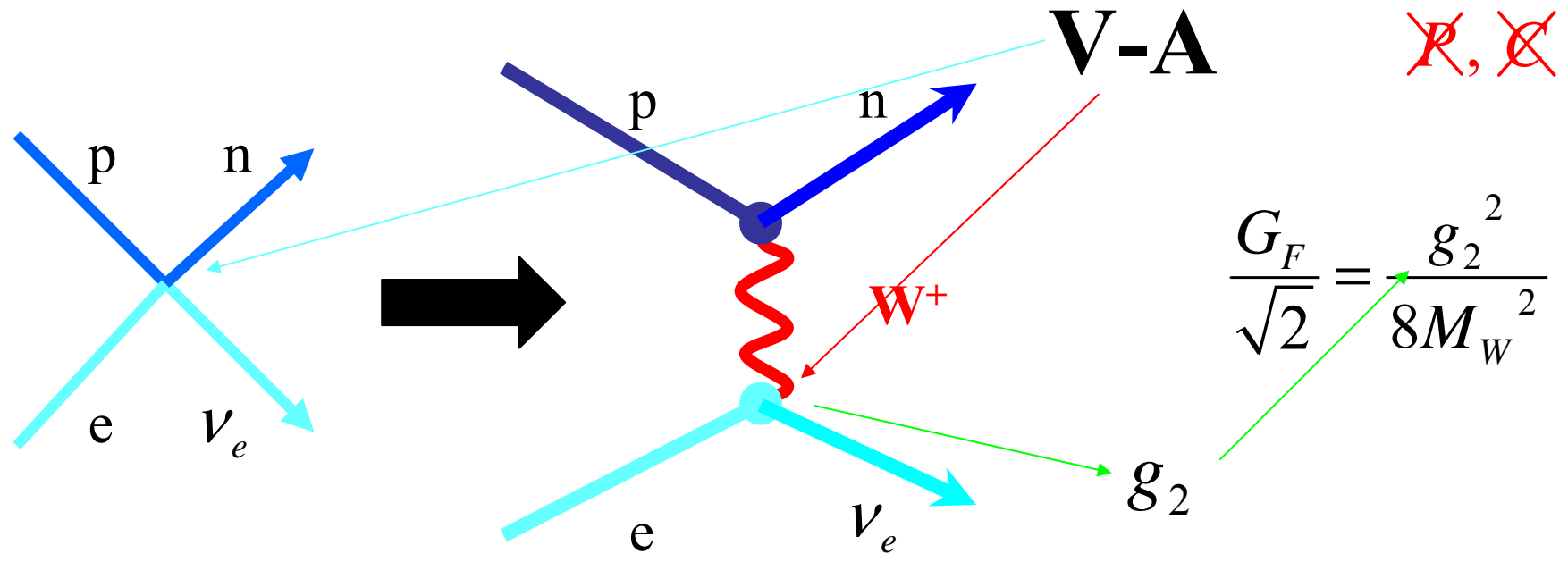
Fermi theory of β decay $n \rightarrow pe \bar{\nu}_e$

$$\frac{G_F}{\sqrt{2}} \left[\bar{u}_n \gamma^\sigma (1 - \gamma_5) u_p \right] \left[\bar{u}_{\nu_e} \gamma_\sigma (1 - \gamma_5) u_e \right]$$

Fermions and the weak interactions

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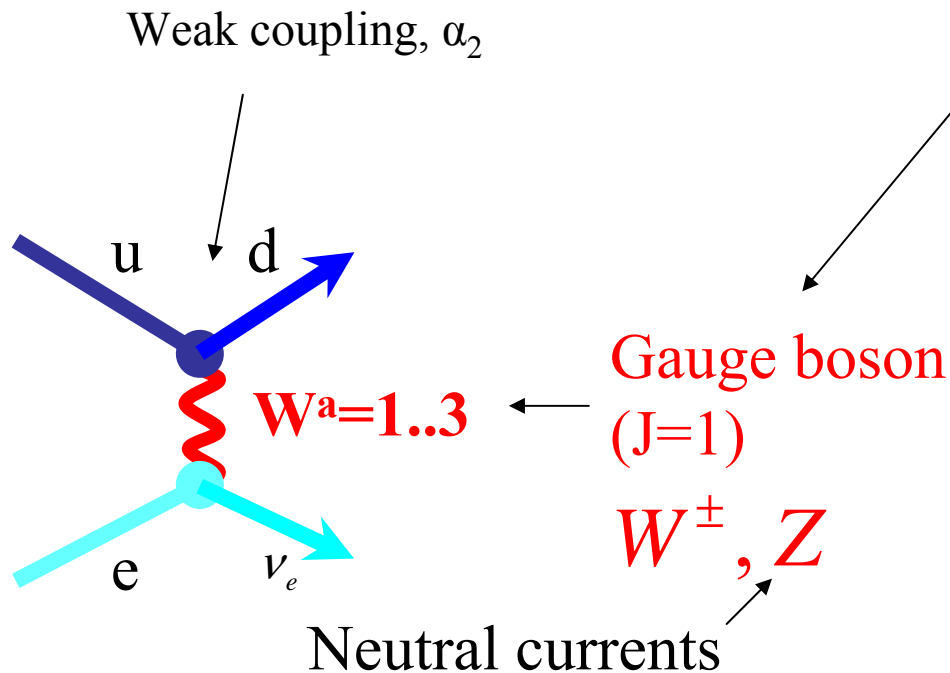


Weak Interactions

SU(2) local gauge theory

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} \nu \\ e \end{pmatrix}_L$$

$$u_R, d_R, e_R$$



Symmetry :

Local conservation of
2 weak isospin charges

$$\Psi_a \rightarrow \left(e^{ig_2 \mathbf{a}(x) \cdot \boldsymbol{\tau}} \right)_b^a \Psi_a$$

$$W_\mu^r \rightarrow W_\mu^r - \partial_\mu \alpha^r - f^{rst} \alpha^s W_\mu^t$$

$$\bar{\Psi} (\partial_\mu - ig_2 \lambda_r W_\mu^r) \gamma^\mu \Psi$$

A non-Abelian (SU(2))
local gauge field theory

Massive vector propagator (W, Z bosons)

$$(g^{\nu\mu}(\partial^2 + M^2) - \partial^\nu \partial_\mu) B^\mu = j^\nu$$

$$(-g^{\mu\nu}(-p^2 + M^2) + p_\mu p_\nu)^{-1} = \frac{i(-g^{\mu\nu} + p^\mu p^\nu / M^2)}{p^2 - M^2}$$

$$B_\mu = \varepsilon_\mu e^{ip \cdot x}$$

$$\varepsilon^{(\lambda=\pm 1)} = \mp(0, 1, \pm i, 0) / \sqrt{2}$$

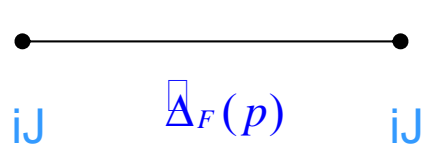
$$\varepsilon^{(\lambda=0)} = (|\mathbf{p}|, 0, 0, E) / M$$

Free particle solution

Helicity polarisation vectors

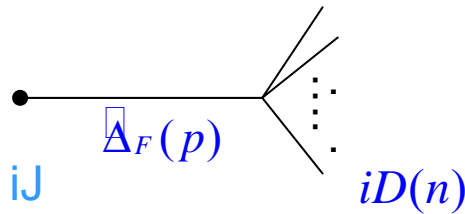
$$\sum_\lambda \varepsilon_\mu^{(\lambda)*} \varepsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}$$

Propagation of **unstable** scalar particle



$$= -iJ^2 \Delta_F(p)$$

No decay



$$= -iJ \Delta_F(p) D(n)$$

Particle decays into final state n

Optical theorem – conservation of probability, time evolution is unitary

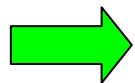
$$S^\dagger S = SS^\dagger = 1$$

$$S_{fi} = \delta_{fi} + iT_{fi}$$

$$\text{Im}(T_{kk}) = \frac{1}{2} \sum_n |T_{nk}|^2$$

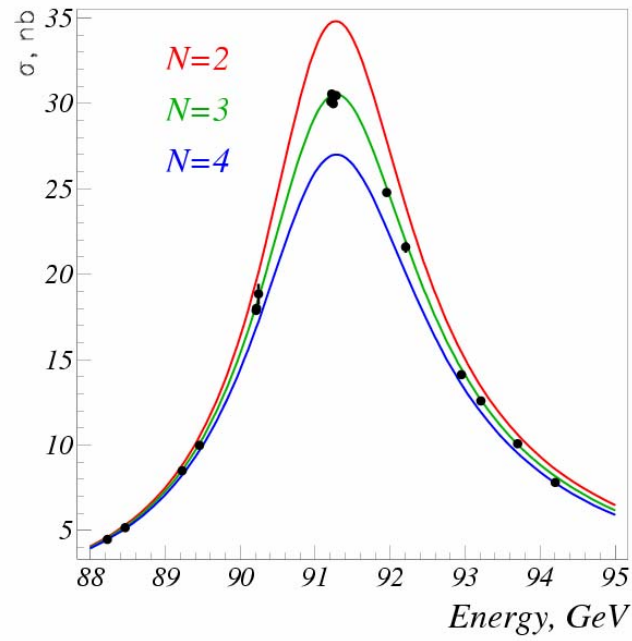
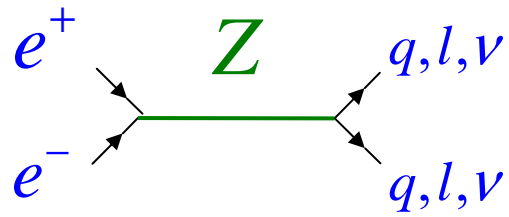
$$\underbrace{m\Gamma_{tot}}$$

$$-J^2 \text{Im}(\Delta_F(p)) = \frac{1}{2} \sum_n \left| -iJ \Delta_F(p) D(n) \right|^2 = J^2 \left| \Delta_F(p) \right|^2 \int \frac{1}{2} \sum_n |D(n)|^2 dQ$$

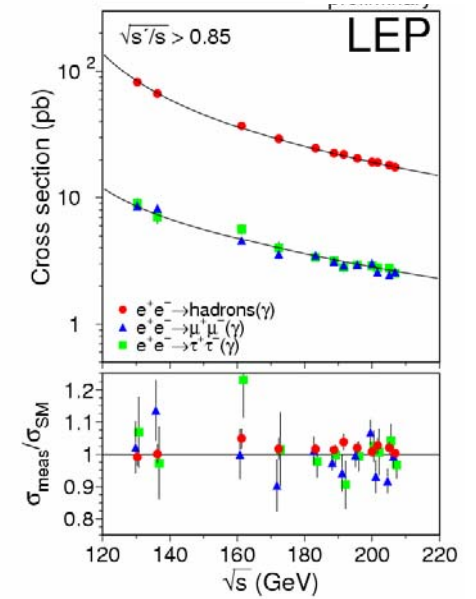


$$\Delta_F(p) = \frac{1}{p^2 - m^2 + im\Gamma_{tot}}$$

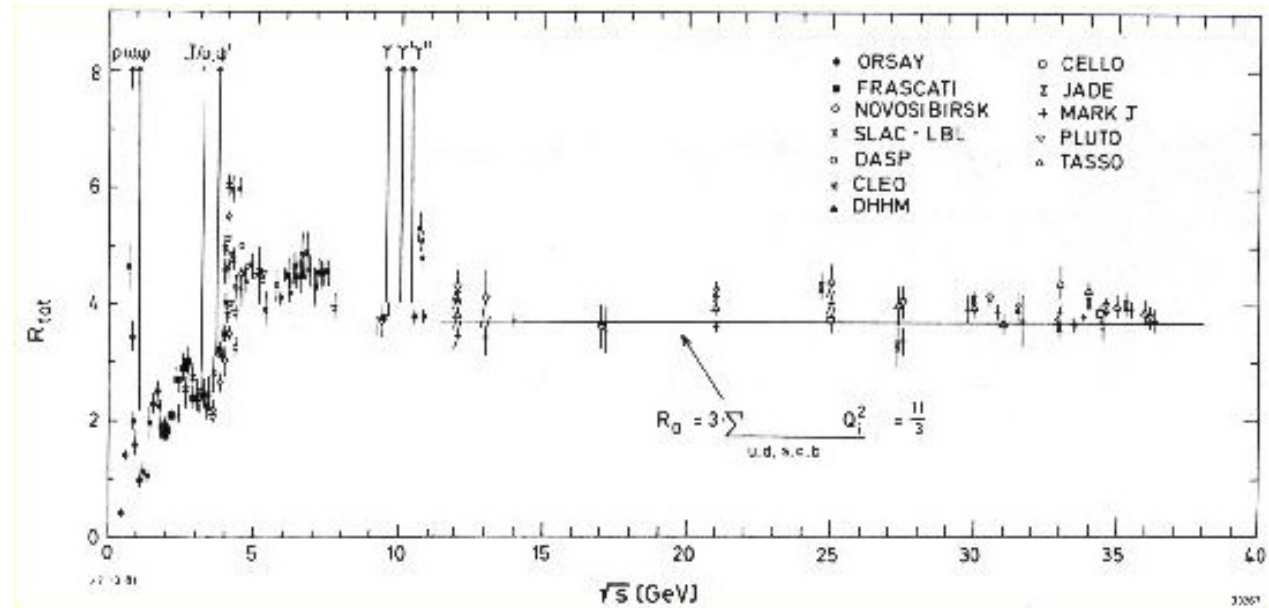
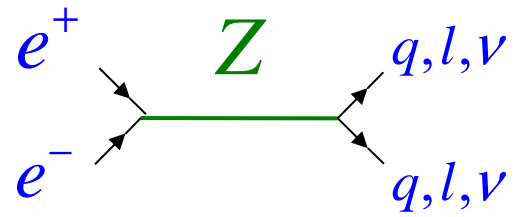
$$(\Delta_F(x) \propto e^{-m\Gamma_{tot} t})$$



(N is no. of light - wrt M_Z - neutrinos)

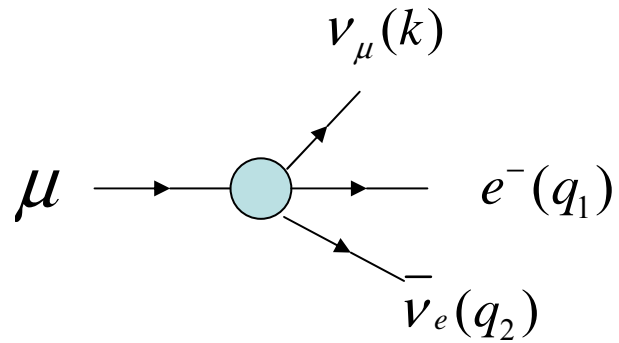


$$\sigma(e^+e^- \rightarrow \text{hadrons}) = \frac{12\pi \Gamma(Z \rightarrow ee)\Gamma(Z \rightarrow \text{hadrons})}{(E^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2}$$



$$\frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 3 \sum_{f=1}^{n_f} Q_f^2$$

μ decay



Fermi theory ('40s)

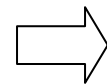
$$M = \frac{G_F}{\sqrt{2}} \bar{u}(k) \gamma^\mu (1 - \gamma_5) u(p) \bar{u}(q) \gamma_\mu (1 - \gamma_5) v(p)$$

Dimensional analysis

$$\Gamma_{tot} = \frac{1}{192\pi^3} m_\mu^5 G_F^2$$

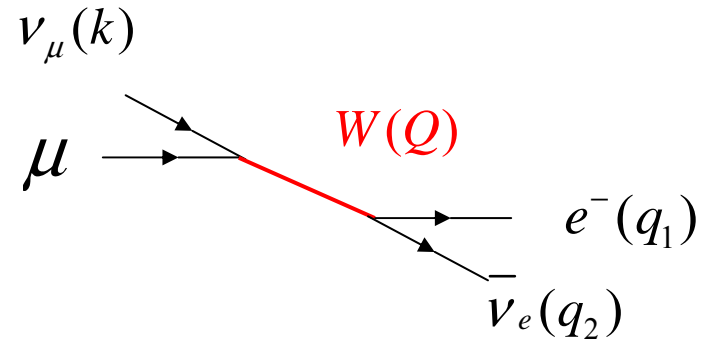
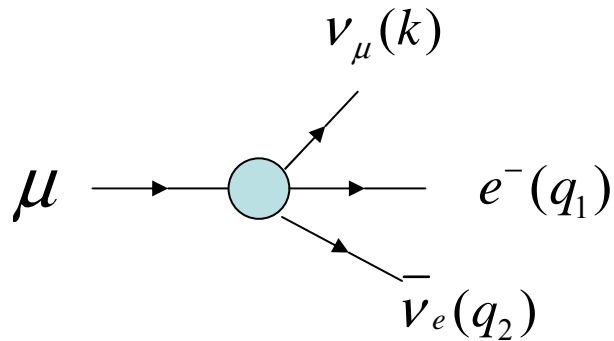
The hard part!

$$\tau_\mu^{\text{expt}} = \frac{1}{\Gamma} = 2.19703(4) 10^{-6} \text{ sec}$$



$$G_F = 1.16637(1) 10^{-5} \text{ GeV}^{-2}$$

μ decay



$$M = ig_W \bar{u}(k) \gamma^\mu (1 - \gamma_5) u(p) \frac{g_{\mu\nu} - \frac{Q_\mu Q_\nu}{M_W^2}}{Q^2 - M_W^2 + i\epsilon} g_W \bar{u}(q) \gamma_\nu (1 - \gamma_5) v(p)$$

$\frac{Q^\mu Q^\nu}{M_W^2} \approx \frac{m_\mu m_e}{M_W^2} \approx 0$

In μ decay $Q^2 \leq O(m_\mu^2) \ll M_W^2$

$$\frac{g_W^2}{Q^2 - M_W^2} \approx \frac{-g_W^2}{M_W^2} \quad \Rightarrow \quad \frac{g_W^2}{M_W^2} = \frac{G_F}{\sqrt{2}}$$

$G_F = 1.16637(1) \cdot 10^{-5} \text{ GeV}^{-2}$
 $\approx 80 \text{ GeV}$

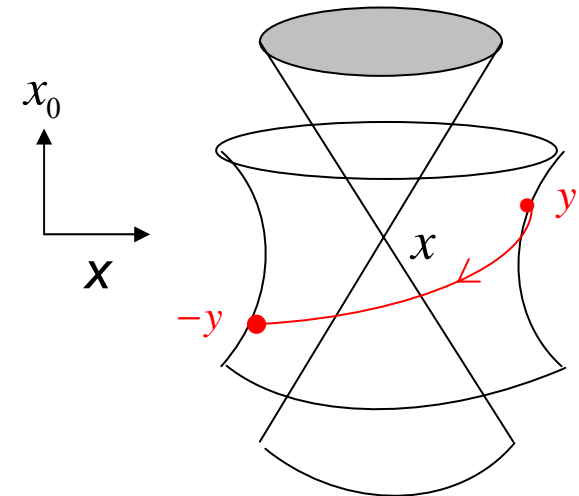
Causality?

QM : $U(x'-x) \propto e^{-m\sqrt{(x'-x)^2 - (t'-t)^2}}$ ✗

See Peskin & Schroeder
"Quantum Field Theory" p28

Field theory :

$$\begin{aligned} \Delta_F(x'-x) &= -i \int \frac{d^3 p}{(2\pi)^3 2\omega_p} e^{-i\omega_p |t'-t| - i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} \\ &= D(x-y) - D(y-x) \end{aligned}$$

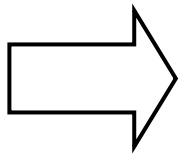


When $(x-y)^2 < 0$, we can perform a Lorentz transformation taking $(x-y) \rightarrow -(x-y)$
...causality preserved $(e^{-m|r|} - e^{-m|r|})$

No (continuous) transformation possible for $(x-y)^2 > 0$
...and amplitude nonvanishing $(e^{-imt} - e^{imt})$

Fundamental principles of particle physics

- Introduction - Fundamental particles and interactions
- Symmetries I - Relativity
- Quantum field theory - Quantum Mechanics + relativity
- Theory confronts experiment - Cross sections and decay rates
- Symmetries II – Gauge symmetries, the Standard Model
- Fermions and the weak interactions



The Standard Model and Beyond

Have Fun!



$$u_{e_L} = \frac{1}{2}(1 - \gamma_5)u_e$$

$$u_{e_R} = \frac{1}{2}(1 + \gamma_5)u_e$$

