THE MYSTERY OF SPINORS

HAMILTON QUATERNIONS

(1843)

DIRAC EQUATION (1928)

PHYSICS

MATHEMATICS

(2006)

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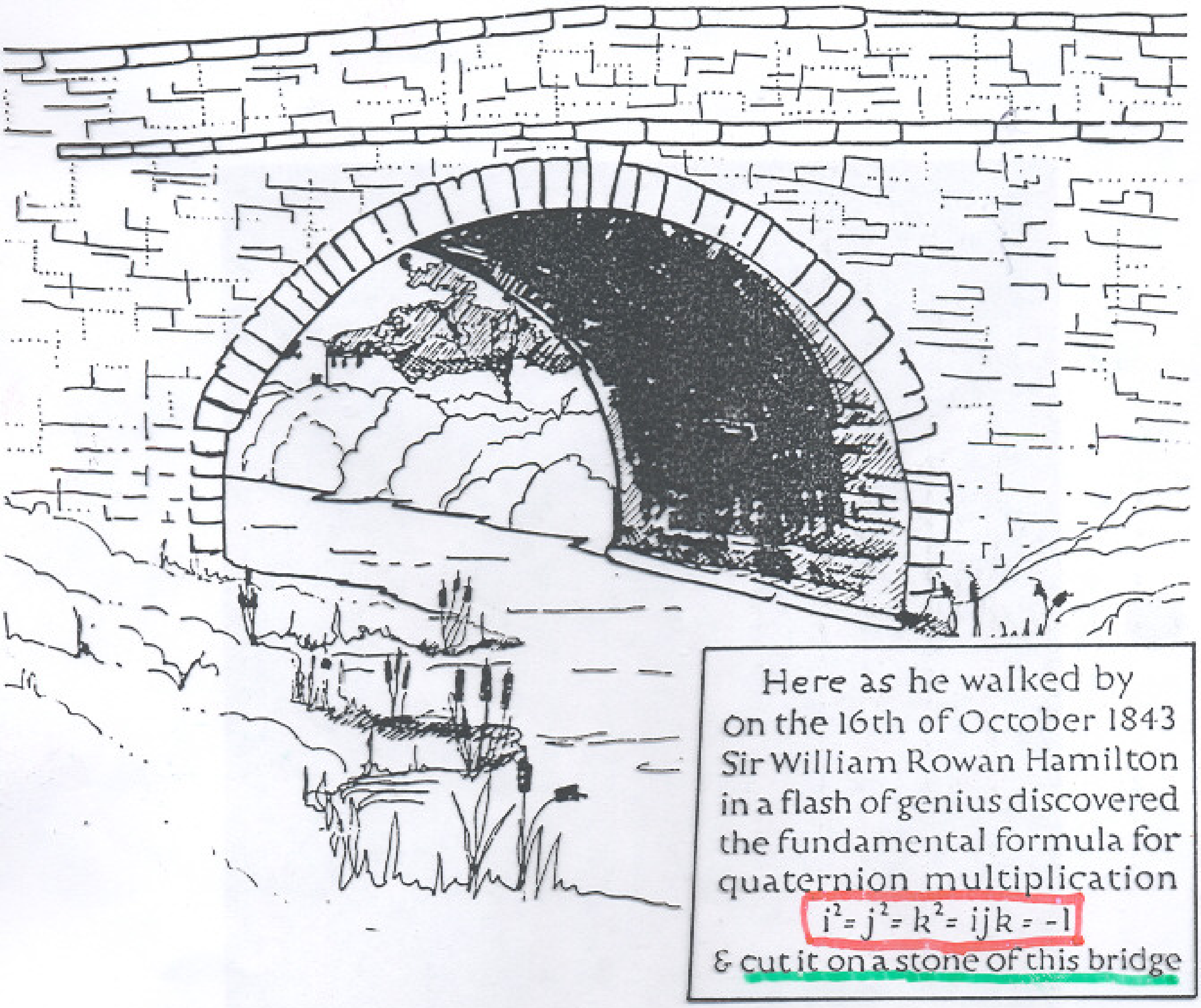
With Compliments

Hamilton in 1842 as President of the Royal Irish Academy

(after The Dublin University Magazine, June 1842)



Brougham Bridge on the Royal Canal — Hamilton's place of enlightenment, with the modern plaque that commemorates his famous discovery of quaternions in 1843 *(Drawing M. Caulfield; Plaque, Courtesy P.A. Wayman, Director, Dunsink Observatory)*

> there felt the galvanic circuit of thought *close*; and the sparks which fell from it were the *fundamental equations between i, j, k*, exactly such as I have used them ever since.
>
> I pulled out, on the spot, a pocket-book which still exists and made an entry on which, *at that very moment*, I felt that it might be worth my while to expend the labour of at least ten (and it might be fifteen) years to come.[27]

One month before his death, Hamilton again recorded the same event for the benefit of his son Archibald and likely future psychologists. The words now are different, but again the account is essentially similar:[28]

> Although (your mother) talked with me now and then, yet an *under-current* of thought was going on in my mind, which gave at last a *result*, whereof it is not too much to say that I felt *at once* an importance.
>
> An *electric* circuit seemed to *close*; and a spark flashed forth, the herald (as I *foresaw, immediately*) of mâny long years to come of definitely directed thought and work ...
>
> Nor could I resist the impulse — unphilosophical as it may have been

QUANTUM
MECHANICS

OPTICS
MECHANICS

HAMILTON

HAMILTONIAN

QUATERNIONS

VECTOR ANALYSIS

NON-COMMUTATIVE
ALGEBRA

cf. EINSTEIN

F2

COMPLEX NUMBERS

LONG HISTORY ... CAUCHY, GAUSS

GAUSS "THE TRUE METAPHYSICS OF $\sqrt{-1}$ IS ELUSIVE"

$$x + iy = r e^{i\theta}$$

ANGULAR PHASE

↓

ABELIAN GAUGE THEORY

ALGEBRA

ANALYSIS , GEOMETRY → ABEL, RIEMANN

↓ (from ANALYSIS)

2 DIM PHYSICS

CAUCHY-RIEMANN EQNS

LAPLACE EQN

↓ (from ABEL, RIEMANN)

ELLIPTIC FUN[illegible]NS

↓

ALGEBRAIC GEOMETRY

HAMILTON WANTED TO GENERALIZE

TO 3-DIMENSIONS <u>FOR PHYSICAL APPLICATIONS</u>

FOUND HE NEEDED <u>4 DIMENSIONS</u>

RELATED TO <u>RELATIVITY</u>

(IMAGINARY <u>TIME</u>)

$\boxed{\text{ALGEBRA AS STUDY OF TIME}}$

QUATERNIONS ARE <u>SPINORS</u>

$q = r u$ $\quad u$ UNIT QUATERNION

3-SPHERE OR <u>SU(2)</u>

<u>NON-ABELIAN PHASE</u>

(LIE GROUP

DIRAC OPERATOR

ALL KEY IDEAS OF MODERN PHYSICS

150 YEARS LATER

[NOT DISJOINT FROM HAMILTONIAN THEORY]

SPINORS AND GEOMETRY

IN 4 DIMENSIONS

SPINORS = QUATERNIONS DIM 4

$$S \otimes S = 1 + 4 + 6 + 4 + 1$$

$(4 \times 4 = 16)$

DIFFERENTIAL FORMS OF ALL DEGREES

GEOMETRY LENGTH, AREA, VOLUME,

$$\text{SPINORS} = \sqrt{\text{GEOMETRY}}$$

WHAT DOES THIS MEAN?

IF $R^4 = C^2$ SPINORS $= 1 + 2 + 1$

COMPLEX GEOMETRY

(PENROSE TWISTOR THEORY)

HODGE THEORY

EUCLIDEAN ANALOGUE OF MAXWELL'S

EQUATIONS (IN VACUO)

$d\omega = 0 \quad d^*\omega = 0 \qquad \omega$ 2-FORM

HODGE (1930-40) GENERALIZED TO

FORMS OF ALL DEGREES p ON

RIEMANNIAN MANIFOLD M

FOR COMPACT M

SOLUTIONS : HARMONIC p-FORMS

$= H^p(M, R)$ COHOMOLOGY

TOPOLOGICAL INDEPENDENT OF METRIC

INTERPRETED BY WITTEN (1982)

AS SUPER-SYMMETRIC QUANTUM MECHANICS

DIRAC OPERATOR

ON COMPACT RIEMANNIAN MANIFOLD M

DIRAC OPERATOR (ACTING ON SPINOR FIELDS) IS ELLIPTIC AND EQN

$$DS = 0 \qquad \text{(HARMONIC SPINORS)}$$

HAS FINITE DIMENSION

BUT VARIES WITH METRIC

FOR $\dim M \geqslant 3$ CAN BE ARBITRARY LARGE

ANY TOPOLOGICAL MEANING ?

IF $\dim M = 2n$ $\quad S = S^+ \oplus S^-$ (CHIRAL SPINORS)

HARMONIC SPINORS $\quad H = H^+ \oplus H^-$

$\dim H^+ - \dim H^-$ TOPOLOGICAL INVARIANT

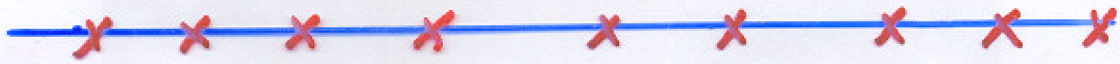
(INDEX D) OF M RELATED TO

ANOMALIES IN QFT (1970's)

SPECTRAL FLOW

dim M ODD D SELF-ADJOINT

SPECTRUM D DISCRETE UNBOUNDED ABOVE AND BELOW



"DIRAC SEA"

COUPLE D WITH GAUGE FIELD VARYING PERIODICALLY IN TIME

FAMILY D_t $0 \le t \le 1$ WITH

D_1 GAUGE-EQUIVALENT TO D_0

SPECTRUM VARIES WITH t BUT ENDS UP SAME

BUT THERE CAN BE A SPECTRAL FLOW OR "SHIFT"

TOPOLOGICAL INVARIANT OF FAMILY D_t

IF $\dim M = 3$

PHYSICAL INTERPRETATION

EXTERNAL FIELD HAS <u>CREATED</u>

(OR <u>ANNIHILATED</u> PARTICLES)

IF WE FORM 4-MANIFOLD

$M \times S^1$

AND GLUE GAUGE-FIELD BY GIVEN

EQUIVALENCE

WE GET ELLIPTIC DIRAC OPERATOR $\mathcal{D}$

ON $M \times S^1$

INDEX $\mathcal{D}$ = SPECTRAL FLOW OF

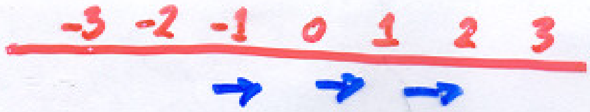
FAMILY D_t

IMAGINARY TIME

PHYSICAL

EXAMPLE - DIMENSION 1

M = CIRCLE



$$D_t = -i\frac{d}{dx} + t \qquad x \bmod 2\pi \quad , \; 0 \le t \le 1$$

$$D_1 = e^{-ix} D_0 e^{ix}$$ GAUGE EQUIVALENCE

$\boxed{\text{SPECTRAL FLOW} = 1}$

ON $M \times S^1$ (TORUS)

$$\mathcal{D} = \frac{\partial}{\partial t} + D_t$$ ACTS ON FUNCTIONS

$f(x,t)$ PERIODIC IN x (PERIOD 2π)

AND SATISFY $f(x, t+1) = e^{-ix} f(x,t)$

THETA-FUNCTION

$$f(x,t) = \sum_n \exp\left(-\frac{(n+t)^2}{2} + inx\right)$$

IS UNIQUE SOLUTION OF $\mathcal{D}f = 0$

NO SOLUTIONS OF $\mathcal{D}^* f = 0$

$\boxed{\text{INDEX } \mathcal{D} = 1}$

DONALDSON THEORY

1983 OPENED UP ENTIRELY

NEW FIELD IN 4-DIMENSIONAL

GEOMETRY USING IDEAS FROM PHYSICS

YANG-MILLS INSTANTONS

SOLUTIONS OF SELF-DUALITY EQNS

$*F = F$

NON-LINEAR HODGE THEORY

ON COMPACT RIEMANNIAN

4-DIMENSIONAL MANIFOLD

DEPENDS ON FINITELY MANY CONTINUOUS

PARAMETERS (MODULI)

MODULI SPACE IS MANIFOLD $\mathcal{M}$

DEPENDS ON METRIC ON M

BUT CERTAIN TOPOLOGICAL INVARIANTS

OF $\mathcal{M}$ INDEPENDENT OF METRIC

⇒ DONALDSON INVARIANTS OF M

VERY UNEXPECTED

DEPEND ON SMOOTH STRUCTURE OF M

VERY POWERFUL

WHAT IS PHYSICAL SIGNIFICANCE?

WITTEN (1988)

EXPLAINED DONALDSON THEORY

AS A PURELY TOPOLOGICAL QFT

N= 2 SUPER-SYMMETRIC (TWISTED)

YANG-MILLS QFT IN 4 DIMENSIONS

SEIBERG - WITTEN THEORY

SEIBERG- WITTEN THEORY IS A

TOP. QFT U(1) -GAUGE THEORY

COUPLED TO SPINORS

EQUIVALENT (DUAL TO)

DONALDSON THEORY

SIMPLER - EASIER TO USE

NOTE NO SPINORS IN DONALDSON

THEORY

MYSTERIOUS

ROLE OF SPINORS ?

DESCRIBE MATTER DIRAC

EXPLAIN POSITIVITY OF MASS WITTEN

RELATED TO SCALAR CURVATURE GEOMETRY

COSMOLOGICAL CONSTANT ?

Quaternions, according to Tait, freed the mathematical physicist from the artificial slavery of coordinates and allowed his thoughts to run in their most natural channels. In a marvelous display of Scottish industrialism and Victorian imperialism he compared coordinates

> to a steam-hammer, which an expert may employ on any destructive or constructive work of *one general kind*, say the cracking of an egg-shell, or the welding of an anchor. But you must have your expert to manage it, for without him it is useless. He has to toil amid the heat, smoke, grime, grease, and perpetual din of the suffocating engine-room. The work has to be brought to the hammer, for it cannot usually be taken to its work. ... Quaternions, on the other hand, are like the elephant's trunk, ready at *any* moment for *anything*, be it to pick up a crumb or a field gun, to strangle a tiger or to uproot a tree. Portable in the extreme, applicable anywhere ... directed by a little native who requires no special skill or training, and who can be transferred from one elephant to another without much hesitation. Surely this, which adapts itself to its work, is the grander instrument! But then, *it* is the natural, the other the artificial, one.[64]

It can be assumed that Tait knew little about elephants or natives, but his feelings about the superiority of quaternions over coordinates are obvious enough.

QUATERNIONS & PHYSICS

HAMILTON 1846

INTRODUCED THE OPERATOR

$$\triangleleft = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

NOTED THE IDENTITY

$$-\triangleleft^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2$$

AND REMARKED THAT

"APPLICATIONS TO ANALYTICAL PHYSICS MUST BE EXTENSIVE TO A HIGH DEGREE"